

MICROLOCAL KZ FUNCTORS AND RATIONAL CHEREDNIK ALGEBRAS.

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ABSTRACT. Following the work of Kashiwara-Rouquier and Gan-Ginzburg, we define a family of exact functors from category \mathcal{O} for the rational Cherednik algebra in type A to representations of certain “coloured braid groups” and calculate the dimensions of the representations thus obtained from standard modules. To show that our constructions also make sense in a more general context, we also briefly study the case of the rational Cherednik algebra corresponding to complex reflection group $\mathbb{Z}/l\mathbb{Z}$.

1. INTRODUCTION

1.1. This paper is inspired by two beautiful constructions in the theory of rational Cherednik algebras of type A . The first is the KZ functor studied in [GGOR] which relates representations in the category \mathcal{O}_c to representations of Hecke algebras, and the second is the localization theorem of Kashiwara-Rouquier (see also the work of Gan and Ginzburg [GG]) which allows us to realize the category \mathcal{O}_c as a category of modules for a “microlocal” sheaf of rings \mathcal{A}_c on the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of n points in the plane (with certain restrictions on the value of the deformation parameter c).

In the context of [KR], the modules in the category \mathcal{O}_c correspond to sheaves of \mathcal{A}_c -modules supported on a certain reducible Lagrangian subvariety Z of $\text{Hilb}^n(\mathbb{C}^2)$ first considered by Grojnowski [Gr]. Using this microlocal point of view, we construct for each component of Z an exact functor from \mathcal{O}_c to a category of local systems, and we show that the KZ -functor naturally corresponds to one of them.

On the level of dimensions, these functors recover the characteristic cycle as defined by Gan and Ginzburg [GG], and for standard modules we are able to calculate these dimensions. This calculation is already known by combining work of Ginzburg, Gordon and Stafford [GGS] with earlier work of Gordon and Stafford [GS], but our calculations are self-contained, and in particular make no use of Haiman’s work on the Hilbert scheme.

In section 2 we recall the definition of the rational Cherednik algebra in type A and the construction of the KZ -functor. In section 3 we review the quantum Hamiltonian reduction construction of Gan-Ginzburg. In section 4 we study the Lagrangian subvariety Z of the Hilbert scheme and exhibit its smooth locus. In section 5 we define for each component Z_λ of Z an exact functor KZ_λ on category \mathcal{O}_c generalizing the KZ -functor. In section 6 we compute the characteristic cycles of standard modules, which give the dimensions of our functors KZ_λ on these modules. In section 7 we show using \mathcal{D} -module techniques that the original KZ -functor descends to a functor to representations of a Hecke algebras (as opposed

to just a braid group). Finally, to give an example of our techniques for a complex reflection group, in section 8 we consider the case of the complex reflection group $\mathbb{Z}/l\mathbb{Z}$. Here we find a direct connection to the theory of cyclotomic q -Schur algebras, while hints of a similar connection to the classical q -Schur algebra are suggested by our results in the case of the symmetric group. We hope to return to this issue in a later paper. Finally in the Appendix we give a review of material on twisted \mathcal{D} -modules and equivariance, and a brief discussion of the index theorem which is needed in our calculation of characteristic cycles of standard modules.

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2. RATIONAL CHEREDNIK ALGEBRA AND THE KZ FUNCTOR.

2.1. Let \mathfrak{h} be the permutation representation of the symmetric group $W = S_n$ on n letters. Thus $\mathfrak{h} = \mathbb{C}^n$ and W acts by permuting the coordinates in the obvious way. The rational Cherednik algebras¹ \mathcal{H}_c of type A , for $c \in \mathbb{C}$, are a family of algebras giving a flat deformation of $\mathcal{D}(\mathfrak{h}) \rtimes \mathbb{C}[W]$, the smash product of the group algebra with the algebra of differential operators on \mathfrak{h} .

Let $\{y_1, y_2, \dots, y_n\}$ be the standard basis of \mathfrak{h} , and let $\{x_1, x_2, \dots, x_n\}$ be the corresponding dual basis of \mathfrak{h}^* . If we let $s_{ij} \in W$ denote the transposition which interchanges i and j , then \mathcal{H}_c is generated by $\{x_i, y_i, s_{ij} : 1 \leq i, j \leq n, i \neq j\}$ subject to relations:

$$\begin{aligned} s_{ij}x_i &= x_js_{ij}, & s_{ij}y_i &= y_js_{ij}, & 1 \leq i, j \leq n, i \neq j, \\ [y_i, x_j] &= cs_{ij}, & [x_i, x_j] &= [y_i, y_j] = 0, & \forall 1 \leq i, j \leq n, i \neq j \\ [y_k, x_k] &= 1 - c \sum_{i \neq k} s_{ik}. \end{aligned}$$

Clearly \mathcal{H}_0 is just $\mathcal{D}(\mathfrak{h}) \rtimes \mathbb{C}[W]$. In general, one may filter \mathcal{H}_c by placing \mathfrak{h} in degree 1, \mathfrak{h}^* and $\mathbb{C}[W]$ in degree 0. The associated graded algebra is then isomorphic to $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes \mathbb{C}[W]$, and the canonical map from \mathcal{H}_c is known to be a vector space isomorphism [EG, Theorem 1.3]. This yields a triangular decomposition for \mathcal{H}_c : multiplication in \mathcal{H}_c induces a vector space isomorphism

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}] \rightarrow \mathcal{H}_c.$$

Motivated by the analogy between this decomposition and the well-known triangular decomposition of a semisimple Lie algebra, one can consider the category $\mathcal{O}_{\mathcal{H}_c}$ of representations for \mathcal{H}_c which are locally finite for the subalgebra $\mathbb{C}[\mathfrak{h}^*]$. This category was introduced in [DO] and has been extensively studied (see for example [BEG], [GGOR]). A module M in category $\mathcal{O}_{\mathcal{H}_c}$ is said to have *type* $\bar{\lambda}$ for $\bar{\lambda} \in \mathfrak{h}^*/W = \text{Spec}(\mathbb{C}[\mathfrak{h}^*]^W)$ if for any element $P \in \mathbb{C}[\mathfrak{h}]^W$ the operator $P - P(\bar{\lambda})$ is locally nilpotent. The full subcategory of $\mathcal{O}_{\mathcal{H}_c}$ consisting of objects of type $\bar{\lambda}$ is denoted $\mathcal{O}_{\mathcal{H}_c}(\bar{\lambda})$ and $\mathcal{O}_{\mathcal{H}_c}$ splits as a direct sum of the subcategories $\mathcal{O}_{\mathcal{H}_c}(\bar{\lambda})$. We shall focus on the category $\mathcal{O}_{\mathcal{H}_c}(0)$, where $\mathbb{C}[\mathfrak{h}]^W$ acts locally nilpotently, and

¹We have set the parameter t of [EG] to be 1.

thus for convenience we shall denote it by \mathcal{O}_c . It is abelian and every object has finite length. Note that the results of Bezrukavnikov and Etingof on induction/restriction functors allow one to reduce the study of $\mathcal{O}_{\mathcal{H}_c}(\bar{\lambda})$ to $\mathcal{O}_{\mathcal{H}'_c}(0)$ for \mathcal{H}'_c a rational Cherednik algebra corresponding to a parabolic subgroup of W , so that questions about $\mathcal{O}_{\mathcal{H}_c}$ can be reduced to ones for $\mathcal{O}_{\mathcal{H}_c}(0)$ (see [BE, Corollary 3.3] for a precise statement).

2.2. In order to relate \mathcal{H}_c to the Hilbert scheme, we will also need its spherical subalgebra, $e\mathcal{H}_ce$, where $e = |W|^{-1} \sum_{w \in W} w$ is the idempotent in $\mathbb{C}[W]$ corresponding to the trivial representation. The map $a \mapsto ae$ yields embeddings from $\mathbb{C}[\mathfrak{h}]^W \rightarrow \mathbb{C}[\mathfrak{h}]^W e \hookrightarrow e\mathcal{H}_ce$ and from $\mathbb{C}[\mathfrak{h}^*]^W \rightarrow \mathbb{C}[\mathfrak{h}^*]^W e \hookrightarrow e\mathcal{H}_ce$. Moreover, if $e\mathcal{H}_ce$ is simple, then their images generate $e\mathcal{H}_ce$ [BEG, Theorem 4.6]. It is easy to see that $e\mathcal{H}_ce$ is simple whenever \mathcal{H}_c is simple², and by [BEG, Theorem 3.1] the algebra \mathcal{H}_c is simple algebra exactly when the associated finite Hecke algebra $H_W(q)$ is semisimple, where $q = e^{2\pi ic}$. One has the obvious functor from \mathcal{H}_c -modules to $e\mathcal{H}_ce$ -modules given by $M \mapsto eM$. This is easily seen [BEG, Lemma 4.1] to be a Morita equivalence when \mathcal{H}_c is simple.

2.3. We now recall the construction of the KZ-functor of [GGOR]. Let

$$\mathfrak{h}^{\text{reg}} = \{v \in \mathfrak{h} : |W \cdot v| = |W|\},$$

be the subset of \mathfrak{h} consisting of those points whose coordinates are pairwise distinct. If we set $\delta = \prod_{i < j} (x_i - x_j) \in \mathbb{C}[\mathfrak{h}]$ then $\mathfrak{h}^{\text{reg}} = \{v \in \mathfrak{h} : \delta(v) \neq 0\}$, hence it is a Zariski-open subset of \mathfrak{h} . The key to the construction is the Dunkl homomorphism from \mathcal{H}_c to $\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}[W]$. This is given by the obvious embedding of the subalgebra $\mathbb{C}[\mathfrak{h}] \rtimes \mathbb{C}[W]$, and by sending $y \in \mathfrak{h}$ to the operator

$$T_y = \partial_y + c \sum_{i < j} \frac{y_i - y_j}{x_i - x_j} (s_{ij} - 1) \in \mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}[W].$$

It is straight-forward to check that $S = \{\delta^k : k \in \mathbb{Z}_{>0}\}$ is an Ore set in \mathcal{H}_c , so that it makes sense to localize \mathcal{H}_c at S . We refer to this as localizing \mathcal{H}_c to $\mathfrak{h}^{\text{reg}}$ and denote the resulting algebra by $\mathcal{H}_{c|\mathfrak{h}^{\text{reg}}}$. Similarly we may localize an \mathcal{H}_c -module at S to obtain an $\mathcal{H}_{c|\mathfrak{h}^{\text{reg}}}$ -module denoted $M|_{\mathfrak{h}^{\text{reg}}}$. We then have the following result:

Proposition 2.3.1. [EG, §4] *The assignment $w \mapsto w$, $x \mapsto x$ and $y \mapsto T_y$ extends to an algebra homomorphism Θ_c which embeds \mathcal{H}_c into $\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}[W]$. Moreover, this map becomes an isomorphism after localizing to $\mathfrak{h}^{\text{reg}}$.*

Note that the final part of the proposition is immediately clear from the first part and the explicit formula for T_y . Now given any module M in the category $\mathcal{O}_{\mathcal{H}_c}$, via the above isomorphism, we may view $M|_{\mathfrak{h}^{\text{reg}}}$ as a module for $\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}[W]$ -module, and hence, since $\mathfrak{h}^{\text{reg}}$ is an affine variety, as a W -equivariant module for the sheaf of differential operators $\mathcal{D}_{\mathfrak{h}^{\text{reg}}}$ on $\mathfrak{h}^{\text{reg}}$. Now it is not hard to show that M is finitely generated as a $\mathbb{C}[\mathfrak{h}]$ -module (see for example Lemma 2.5(i) of [BEG]), thus $M|_{\mathfrak{h}^{\text{reg}}}$ is coherent as an $\mathcal{O}_{\mathfrak{h}^{\text{reg}}}$ -module. It follows that the associated $\mathcal{D}_{\mathfrak{h}^{\text{reg}}}$ -module must be a vector bundle with a flat connection, and hence it gives rise to a local system on $\mathfrak{h}^{\text{reg}}$. Since this local system is W -equivariant, it descends to give a local

²However it is also shown in [BE, Theorem 4.10] that $e\mathcal{H}_ce$ is simple for all $c \in (-1, 0)$, a range in which \mathcal{H}_c is not always simple.

system on $\mathfrak{h}^{\text{reg}}/W$. But now if $x_0 \in \mathfrak{h}^{\text{reg}}$ is any point, it is known that $\pi_1(\mathfrak{h}^{\text{reg}}/W, x_0)$ is isomorphic to the braid group \mathcal{B}_W , and hence we get a functor

$$\text{KZ}: \mathcal{O}_{\mathcal{H}_c} \rightarrow \text{Rep}(\mathcal{B}_W),$$

by for example applying the deRham functor to $M_{\mathfrak{h}^{\text{reg}}}$ (alternatively one could take the dual representation given by the solution complex). One of the main results of [GGOR] shows that the image of this functor lies in the much smaller category of representations of the associated Hecke algebra.

Theorem 2.3.1. ([GGOR, Theorem 5.13]) *The functor KZ factors through the category of representations of the Hecke algebra \mathcal{H}_q where the parameter q is specialized to $e^{2\pi ic}$.*

We will give a new proof of this theorem in Section 5 in the \mathcal{D} -module context of [GG].

3. THE DEFORMED HARISH-CHANDRA HOMOMORPHISM AND THE HILBERT SCHEME.

3.1. In this section we recall the localization of the rational Cherednik algebra in type A , as studied in [KR] and the related work [GG]. Let V be an n -dimensional vector space and let $\mathfrak{g} = \mathfrak{gl}(V)$ be the Lie algebra of linear maps from V to itself. The almost commuting variety is the variety of quadruples

$$\mathcal{M} = \{(X, Y, i, j) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^* : [X, Y] + ji = 0\},$$

where ji denotes the rank one linear map $v \mapsto j(v).i$. Using the trace form we may naturally identify \mathfrak{g} with \mathfrak{g}^* , so that \mathcal{M} may be viewed as a subvariety of $T^*\mathfrak{G}$ where we set $\mathfrak{G} = \mathfrak{g} \times V$. It is known [GG, Theorem 1.1] that \mathcal{M} is an equidimensional variety with $n+1$ components of dimension $n^2 + 2n$. We also consider the set

$$\mathcal{M}_{\text{nil}} = \{(X, Y, i, j) \in \mathcal{M} : Y \text{ is nilpotent}\}.$$

This is a (reducible) Lagrangian subvariety. Finally we consider the Zariski open set (given by a stability condition)

$$\mathcal{U} = \{(X, Y, i, j) \in T^*\mathfrak{G} : \mathbb{C}\langle X, Y \rangle i = V\}$$

where $\mathbb{C}\langle X, Y \rangle$ is the subalgebra of $\text{End}(V)$ generated by X and Y . If we let $\mathcal{M}^s = \mathcal{M} \cap \mathcal{U}$, then it can be shown that $\text{GL}(V)$ acts freely on \mathcal{M}^s , and the quotient is isomorphic to $\text{Hilb}^n(\mathbb{C}^2)$, the Hilbert scheme of n points in the plane. At the level of sets the isomorphism can be seen as follows: if $(X, Y, i, j) \in \mathcal{M}^s$ then the stability condition in fact forces j to be zero, so that X and Y commute. Thus they define a homomorphism from $\mathbb{C}[x, y] \rightarrow \text{End}(V)$, and the kernel of this homomorphism is a codimension n ideal of $\mathbb{C}[x, y]$. Note that this realization of $\text{Hilb}^n(\mathbb{C}^2)$ equips it with a natural ample line bundle corresponding to the G -equivariant line bundle $\mathcal{M}^s \times \det$ on \mathcal{M}^s . Let \mathcal{M}^{sn} denotes the intersection of \mathcal{M}^s with \mathcal{M}_{nil} . It is G -stable its quotient is a Lagrangian subvariety of $\text{Hilb}^n(\mathbb{C}^2)$ which we denote by Z .

3.2. We now relate the rational Cherednik algebra to twisted \mathcal{D} -modules. For the convenience of the reader, and to fix notation, in Appendix 9.1 we review a construction of the kinds of twisted rings of differential operators we need, along with notions of equivariance. Let $V^\circ = V \setminus \{0\}$, let $\mathfrak{G}^\circ = \mathfrak{g} \times V^\circ$, and let $\mathfrak{X} = \mathfrak{g} \times \mathbb{P}$ where $\mathbb{P} = \mathbb{P}(V)$ is the projective space of lines in V . Clearly \mathfrak{G}° is a principal

\mathbb{G}_m -bundle over \mathfrak{X} , so that given any $\lambda \in \text{Hom}(\text{Lie}(\mathbb{G}_m), \mathbb{C})$, we may consider modules for the corresponding sheaf $\mathcal{D}_{\mathfrak{X}, \lambda}$ of λ -twisted differential operators on \mathfrak{X} , or equivalently (\mathbb{G}_m, λ) -twisted equivariant $\mathcal{D}_{\mathfrak{G}^\circ}$ -modules. Note that we may describe the cotangent bundle of \mathfrak{X} as

$$\{(X, Y, i, j) \in T^*\mathfrak{G}^\circ : \langle i, j \rangle = 0\} / \mathbb{C}^\times,$$

where \mathbb{C}^\times acts via $t \cdot (X, Y, i, j) = (X, Y, ti, t^{-1}j)$ and as above we have identified $T^*\mathfrak{G}^\circ$ with $\mathfrak{g} \times \mathfrak{g} \times V^\circ \times V^*$. The function $(X, Y, i, j) \mapsto [X, Y] + ij$ is fixed by this \mathbb{C}^\times -action, and moreover if $(X, Y, i, j) \in \mathcal{M}$ then

$$\langle i, j \rangle = \text{Tr}(ji) = -\text{Tr}([X, Y]) = 0,$$

thus

$$\mathcal{M} \cap \{(X, Y, i, j) \in T^*\mathfrak{G}^\circ : \langle i, j \rangle = 0\} = \mathcal{M} \cap T^*\mathfrak{G}^\circ.$$

Let \mathfrak{M} denote the quotient of $\mathcal{M} \cap T^*\mathfrak{G}^\circ$ by \mathbb{C}^\times , a subvariety of $T^*\mathfrak{X}$, and similarly let Λ be the quotient of $\mathcal{M}_{\text{nil}} \cap T^*\mathfrak{G}^\circ$ (a Lagrangian subvariety of $T^*\mathfrak{X}$). If $(X, Y, i, j) \in \mathcal{M}^s$ then $i \neq 0$ and $\langle i, j \rangle = 0$ (since $j = 0$), so that $\mathcal{M} \cap T^*\mathfrak{G}^\circ$ contains \mathcal{M}^s and \mathcal{M}^{sn} . We denote their quotients in $T^*\mathfrak{X}$ by \mathfrak{M}^s and Λ^s .

The action of $\text{SL}(V)$ on \mathfrak{G}° commutes with the \mathbb{G}_m -action, so that we may also consider the category of $\text{SL}(V)$ -equivariant $\mathcal{D}_{\mathfrak{X}, \lambda}$ -modules. Now if we write $\mathcal{Z} \cong \mathbb{G}_m$ for the centre of $\text{GL}(V)$, then the action of \mathbb{G}_m on \mathfrak{G}° is clearly the restriction to \mathcal{Z} of the action of $\text{GL}(V)$ on \mathfrak{G}° . We may thus also consider the category of $(\text{GL}(V), c.\text{tr})$ -equivariant \mathcal{D} -modules on \mathfrak{G}° where as usual tr denotes the trace character on \mathfrak{g} . Note that given $c \in \mathbb{C}$, the corresponding λ above for $\text{Lie}(\mathbb{G}_m) = \text{Lie}(\mathcal{Z})$ is given by $\lambda = c.\text{tr}|_{\text{Lie}(\mathcal{Z})}$, and following [GGS] we will from now on write $\mathcal{D}_{\mathfrak{X}, c}$ for $\mathcal{D}_{\mathfrak{X}, \lambda}$. Now since tr vanishes on $\mathfrak{sl}(V)$ and $\text{GL}(V) = \mathcal{Z}.\text{SL}(V)$, this category is equivalent to the category of $\text{SL}(V)$ -equivariant $\mathcal{D}_{\mathfrak{X}, c}$ -modules on \mathfrak{X} and moreover this equivalence respects holonomic modules. Note that the category of $(\text{GL}(V), c.\text{tr})$ -equivariant modules is considered in [KR], while the category of $\text{SL}(V)$ -equivariant $\mathcal{D}_{\mathfrak{X}, c}$ -modules is used in [GG]. For more details concerning this equivalence, at least at the level of algebras of global sections, see [GGS, §6].

We also remark that if we write $\mathcal{D}_c(\mathfrak{X})$ for the global sections of $\mathcal{D}_{\mathfrak{X}, c}$, then provided $n > 3$ or $n = 2$ and $2c \notin \mathbb{Z}_{\leq 0}$, it is shown in [GGS, §6.2] that

$$\mathcal{D}_c(\mathfrak{X}) \cong (\mathcal{D}(\mathfrak{G}) / (\mathcal{D}(\mathfrak{G})(\nu - c.\text{tr})(\mathfrak{z})))^{\mathcal{Z}}$$

where $\nu: \mathfrak{gl}(V) \rightarrow \Theta_{\mathfrak{g} \times V}$ is the quantized moment map for the $\text{GL}(V)$ -action, and \mathfrak{z} is the Lie algebra of \mathcal{Z} . Thus one may effectively work on \mathfrak{G} or \mathfrak{G}° .

3.3. The action of $\text{SL}(V)$ on \mathfrak{X} induces an infinitesimal action of \mathfrak{sl}_n , and hence a homomorphism from \mathfrak{sl}_n to $\Theta_{\mathfrak{X}}$ the Lie algebra of vector fields on \mathfrak{X} , which extends naturally to a map $\tau: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}_{\mathfrak{X}}$. Similarly, we obtain a homomorphism $\tau_c: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}_{\mathfrak{X}, c}$ from the enveloping algebra to the sheaf of c -twisted differential operators. Let \mathfrak{g}_c denote the image of $\mathfrak{sl}(V)$ under τ_c . Since projective space is known to be \mathcal{D} -affine for $c \notin \{-k/n : k \in \mathbb{Z}_{>0}\}$, for these values the category of $\mathcal{D}_{\mathfrak{X}, c}$ -modules is equivalent to the category of modules for its algebra of global sections $\mathcal{D}_c(\mathfrak{X})$.

One of the main results of [GG] is the construction of a “deformed Harish-Chandra homomorphism”

$$\Phi_c: (\mathcal{D}_c(\mathfrak{X}) / \mathcal{D}_c(\mathfrak{X}) \cdot \mathfrak{g}_c)^{\text{ad}(\mathfrak{g}_c)} \rightarrow e\mathcal{H}_ce.$$

The map Φ_c is a filtered isomorphism sending $\mathbb{C}[\mathfrak{g}]^{\text{Ad}G}$ to $\mathbb{C}[\mathfrak{h}]^{S_n}$ and \mathfrak{Z} , the algebra of constant coefficient differential operators on \mathfrak{g} , to $\text{Sym}(\mathfrak{h})^{S_n}$ (see Theorem 1.5 of [GG]).

Since we will later examine the connection between the original KZ -functor and our microlocal functors, we review briefly the construction of the map Φ_c . It consists of two steps: first there is the Dunkl embedding which we have already discussed. This gives a map $\Theta_c: \mathcal{H}_c \rightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}[W]$. On the other hand one can construct a “radial parts” homomorphism

$$\mathfrak{R}_c: (\mathcal{D}_c(\mathfrak{X})/\mathcal{D}_c(\mathfrak{X}) \cdot \mathfrak{g}_c)^{\text{ad}(\mathfrak{g}_c)} \rightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}})^W.$$

To construct this map let $\mathfrak{X}^{\text{reg}}$ be the open subset of \mathfrak{X} consisting of pairs (X, ℓ) where the nonzero vectors in ℓ are cyclic for X and let $\mathfrak{G}^{\text{reg}}$ denote its preimage in \mathfrak{G}° . If we pick a holomorphic volume form $\Omega \in \bigwedge^n(V^*)$ then the function

$$s: \mathfrak{G} \rightarrow \mathbb{C}, \quad (X, v) \mapsto \langle v \wedge Xv \wedge \dots \wedge X^{n-1}v, \Omega \rangle,$$

shows that $\mathfrak{G}^{\text{reg}}$ is a principal affine open of \mathfrak{G} , and hence $\mathfrak{X}^{\text{reg}}$ is the complement of a principal divisor in \mathfrak{X} . It is easy to check that the action of $\text{PGL}(V)$ on $\mathfrak{X}^{\text{reg}}$ is free (see [BFG, §5.3] or Proposition 4.4.1 below), making it into a principal $\text{PGL}(V)$ -bundle over \mathfrak{h}/W . Thus by descent we see that

$$(\mathcal{D}_c(\mathfrak{X}^{\text{reg}})/\mathcal{D}_c(\mathfrak{X}^{\text{reg}}) \cdot \mathfrak{g}_c)^{\text{PGL}(V)} \cong \mathcal{D}(\mathfrak{h}/W), \quad (c \in \mathbb{C}).$$

where the isomorphism is given explicitly by

$$\mathfrak{R}_c(D) = s^c(D|_{\mathcal{O}(\mathfrak{X}^{\text{reg}, c})})s^{-c}, \quad D \in \mathcal{D}(\mathfrak{X})^G.$$

Now we have inclusions $\mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}/W) \subset \mathcal{D}(\mathfrak{h}^{\text{reg}})^W$, and it can be shown that the image in $\mathcal{D}(\mathfrak{h}^{\text{reg}})^W$ of the composite of \mathfrak{R}_c with this inclusion is precisely the image of the spherical subalgebra of \mathcal{H}_{c-1} under the Dunkl homomorphism. A careful discussion of this fact is given in the Appendix to [GGS]. However the map Ψ_c which we work with is, as in [BFG, §5.4], a twist of this radial parts map by the discriminant δ : we set

$$\Psi_c(D) = m_\delta^{-1} \circ \mathfrak{R}_c(D) \circ m_\delta,$$

where m_f denotes the operator of multiplication by the function f . The image $\text{im}(\Psi_c)$ of this twisted radial parts map is exactly the image of $e\mathcal{H}_c e$ under the Dunkl homomorphism, and the composition of Ψ_c with the inverse of the Dunkl map yields the required isomorphism Φ_c . Note that when $c = 0$ this then agrees with the classical Harish-Chandra homomorphism [EG, §7].

Remark 3.3.1. A similar result is obtained in [KR, Lemma 4.7].

4. ON THE SMOOTH LOCUS OF Λ^s

4.1. In this section we study the Lagrangian variety Z in $\text{Hilb}_n(\mathbb{C}^2)$ introduced in §3.1. Let us first give another description of Z , following [Gr]. Let $\pi: \text{Hilb}^n(\mathbb{C}^2) \rightarrow S^n(\mathbb{C}^2)$ be the Hilbert-Chow morphism. It is known that π is a resolution of singularities. Let $\Sigma = \mathbb{C}$, and view \mathbb{C}^2 as $T^*\Sigma$, a local model for a curve inside a surface. Since $\text{Hilb}^n(\Sigma)$ is isomorphic to the n -th symmetric product of Σ , we may view $\text{Hilb}^n(\Sigma)$ as either a subvariety of $\text{Hilb}^n(\mathbb{C}^2)$ or $S^n(\mathbb{C}^2)$, compatibly with the morphism π . The variety $S^n(\Sigma) = \mathbb{C}^n/S_n$ is stratified in an obvious way:

$$S^n(\Sigma) = \bigsqcup_{\lambda \vdash n} S_\lambda^n(\Sigma),$$

where the stratum $S_\lambda^n(\Sigma)$ contains those points of $S^n(\Sigma)$ whose multiplicities are given by λ , a partition of n . If we set $Z_\lambda^0 = \pi^{-1}(S_\lambda^n(\Sigma))$, then

$$Z = \bigsqcup_{\lambda \vdash n} Z_\lambda^0,$$

is a closed Lagrangian subvariety of $\text{Hilb}^n(\mathbb{C}^2)$, with components $Z_\lambda = \overline{Z_\lambda^0}$. In terms of the analogy with category \mathcal{O} for semisimple Lie algebras, Z corresponds to the union of the conormal varieties to the B -orbits on the flag variety.

$\text{Hilb}^n(T^*\Sigma)$ inherits a natural \mathbb{C}^\times -action from the \mathbb{C}^\times -action on the fibres of $T^*\Sigma$, and the components of Z may also be described in terms of this action. Indeed each component of the fixed point set can naturally be identified with $S_\lambda^n(\Sigma)$ for a partition λ of n , and if Fix_λ denotes this component, we have

$$\overline{Z_\lambda^0} = \overline{\{x \in \text{Hilb}^n(\mathbb{C}^2) : \lim_{t \rightarrow 0} t \cdot x \in \text{Fix}_\lambda\}},$$

(here the right-hand side asserts both that the limit exists, and that it lies in Fix_λ). For more details see [Na, §7.2]

4.2. We now relate this to the almost commuting variety and the construction of $\text{Hilb}^n(\mathbb{C}^2)$ given in the previous section. The variety \mathcal{M}_{nil} is described in [GG] explicitly in terms of a stratification of \mathfrak{X} which mimics Lusztig's stratification of a reductive group or Lie algebra. The strata are labelled by conjugacy classes of pairs (\mathfrak{l}, Ω) where \mathfrak{l} is a Levi subalgebra and Ω is an orbit of L (the associated Levi subgroup) on $\mathcal{N}_\mathfrak{l} \times V$ where $\mathcal{N}_\mathfrak{l}$ is the nilpotent cone of \mathfrak{l} (there are only finitely many such orbits). A stratum S is said to be *relevant* if for some (and hence any) $(X, \ell) \in S$, the element X is regular, and the subspace $\mathbb{C}[X]\ell$ has a $\mathbb{C}[X]$ -invariant complement. Then we have the following result.

Theorem 4.2.1. [GG, §4.3] *The variety Λ is given by*

$$\Lambda = \bigsqcup_{S \text{ relevant}} \overline{T_S^*(\mathfrak{X})}.$$

Now Z is obtained from Λ as the quotient of Λ^s by the $\text{PGL}(V)$ -action, thus the components of Z correspond to certain components of Λ . To make this correspondence explicit, we first give a natural labelling of the components of Λ . Recall that a bipartition of n is an ordered pair of partitions (λ, μ) such that $|\lambda| + |\mu| = n$, where for a partition $\nu = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_k)$ we write $|\nu| = \sum_{i=1}^k \nu_i$, the sum of the parts of ν .

Lemma 4.2.1. *The relevant strata of \mathfrak{X} and hence the components of Λ are labelled by bipartitions of n .*

Proof. Clearly we need only check that we may index relevant strata this way. If $(X, \ell) \in S$ where S is a relevant stratum, then we may write $X = z + x$ the Jordan decomposition of X , where z is semisimple and x is nilpotent. The dimensions of the eigenspaces of z give a partition ν of n (this corresponds to the conjugacy class of the Levi subalgebra $Z_{\mathfrak{g}}(z)$). Moreover, the condition that (X, ℓ) is relevant implies that if $v \in \ell$ is a nonzero vector, then its projection to an eigenspace of z is either zero, or a cyclic vector for x restricted to that subspace. Dividing ν according to this dichotomy we obtain a bipartition as required. \square

Definition 4.2.1. For a bipartition (λ, μ) we will write $\Lambda_{\lambda, \mu}$ for the component of Λ corresponding to (X, ℓ) where the partition λ measures the dimensions of the generalized eigenspaces of X in which the projection of ℓ is nonzero, while μ measures the dimensions of the generalized eigenspaces in which the projection of ℓ is zero.

To relate this to the variety Z we need to examine which components of Λ intersect the stable locus \mathfrak{M}^s . For a bipartition (λ, μ) of n let $S_{\lambda, \mu}$ denote the corresponding relevant stratum and $\Lambda_{\lambda, \mu} = \overline{T_{S_{\lambda, \mu}}^* \mathfrak{X}}$ denote the conormal variety of the stratum labelled by (λ, μ) by the above convention.

We now note the following easy lemma, which is presumably well-known.

Lemma 4.2.2. *Let $X \in \mathfrak{gl}(V)$ be a regular element. Then*

$$Z_{\mathfrak{g}}(X) = \mathbb{C}[X],$$

that is, the matrices which commute with X are precisely the subalgebra of $\text{End}(V)$ generated by X . Moreover it follows that the elements of $Z_{\mathfrak{g}}(X)$ which are nilpotent form a linear subspace.

Proof. Let $V = \bigoplus_{i=1}^k V_i$ be the decomposition of V into the generalized eigenspaces of X . On each V_i we have $X|_{V_i} = \lambda_i + N_i$, where λ_i is a scalar and N_i is a regular nilpotent endomorphism (since X is regular). The standard proof of the existence of a Jordan decomposition (e.g. by the Chinese remainder theorem) shows that the projection operators π_i from V to the generalized eigenspaces of X lie in $\mathbb{C}[X]$ (in fact they are polynomials in X with no constant term). Thus $\iota_i \circ N_i \circ \pi_i$ also lies in $\mathbb{C}[X]$ for each i , ($1 \leq i \leq k$) where ι_i is the inclusion of V_i into V . It is easy to check that the endomorphisms of a vector space U which commute with a regular nilpotent N are exactly the linear maps in the subalgebra $\mathbb{C}[N]$, and since $\{1, N, N^2, \dots, N^{\dim(U)-1}\}$ clearly form a basis of $\mathbb{C}[N]$ this is a vector space of dimension $\dim(U)$. The nilpotent such endomorphisms are simply those in the span of $\{N, N^2, \dots\}$.

If Y commutes with X , then clearly it preserves the eigenspaces of X , and so we may write it as $Y = \sum_{i=1}^k Y_i$, where $Y_i = \iota_i \circ y_i \circ \pi_i$ with $y_i \in \text{End}(V_i)$. Since y_i clearly commutes with $X|_{V_i}$ and hence with N_i we see $y_i \in \mathbb{C}[N_i]$ and so each $Y_i \in \mathbb{C}[X]$, and hence $Y \in \mathbb{C}[X]$. Moreover the nilpotent such Y again clearly form a linear subspace as required. \square

Now we have the following:

Lemma 4.2.3. *The intersection $\mathfrak{M}^s \cap \Lambda_{\lambda, \mu}$ is dense in $\Lambda_{\lambda, \mu}$ if and only if $\mu = \emptyset$.*

Proof. Let $S = S_{\lambda, \mu}$. We will in fact we show that

$$\mathfrak{M}^s \cap T_S^*(\mathfrak{X}) = \begin{cases} \emptyset & \text{if } \mu \neq \emptyset, \\ T_S^*(\mathfrak{X}) & \text{if } \mu = \emptyset. \end{cases}$$

Suppose that (X, Y, i, j) is a point in the intersection, and let $X = z + x$ be the Jordan decomposition of X , where z is semisimple and x is nilpotent. Let $V = \bigoplus_{i=1}^k V_i$ be the decomposition of V according to the eigenspaces of z . The stability condition forces j to be zero, so that X and Y commute, and hence Y commutes with both z and x . As Y and x commute with z , they both preserve the subspaces V_i and we write x_i for $x|_{V_i}$ and y_i for $Y|_{V_i}$ respectively ($1 \leq i \leq k$).

Since $(X, \mathbb{C} \cdot i)$ is relevant, x_i must be a regular nilpotent, and so its centralizer in $\text{End}(V_i)$ is just $\mathbb{C}[x_i]$. Thus $y_i \in \mathbb{C}[x_i]$. Since the projections $V \rightarrow V_i$ also lie in $\mathbb{C}[X]$ it follows that y_i (or rather its extension by zero to V) lies in $\mathbb{C}[X]$ and hence $Y \in \mathbb{C}[X]$. But then $\langle X, Y \rangle = \mathbb{C}[X]$, and the stability condition then forces i to be a cyclic vector for X , and hence $\mu = \emptyset$ as required. Conversely if $\mu = \emptyset$, then it is evident that any point in $T_{\mathfrak{S}}^*(\mathfrak{X})$ satisfies the stability condition. \square

Remark 4.2.1. Note that the above proof does *not* imply that $\Lambda_{\lambda, \emptyset} \cap \mathfrak{M}^s$ is contained in $T_{S_{\lambda, \emptyset}}^*(\mathfrak{X})$. For example when the bipartition is $((n), \emptyset)$, then the points of the corresponding stratum consist of matrices of the form $c + n$ where c is a scalar and n is a regular nilpotent, together with a line which is cyclic for n . The closure of the corresponding conormal variety contains stable quadruples $(X, Y, i, 0)$ where X is not regular. If we take $n = 3$ say then it is easy to see that

$$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, 0 \right),$$

is a stable quadruple, with neither X or Y regular. On the other hand, since U is Zariski open, so is $(U \cap T^*\mathfrak{G}^\circ)/\mathbb{C}^\times$, thus if a component of Λ intersects \mathfrak{M}^s it will do so in a dense open subset. It follows that the Lemma tell us exactly which components of Λ have nontrivial intersection with Λ^s .

Recall we set $\mathcal{M}^{\text{sn}} = \mathcal{M}^s \cap \mathcal{M}_{\text{nil}}$. It is $\text{GL}(V)$ -stable. Its quotient by $\text{GL}(V)$ is the variety Z , and its quotient by the action of \mathcal{Z} is the variety $\Lambda^s = \Lambda \cap \mathfrak{M}^s$. From Lemma 4.2.3 and Remark 4.2.1 we see that the components of Λ^s are labelled by bipartitions of n of the form (λ, \emptyset) . Since for a point $(X, Y, \ell) \in \Lambda^s$, the spectrum of X yields the n -tuple of points in $S^n(\Sigma)$ under the composition of the quotient map with the Hilbert-Chow morphism, it is easy to check that the component Z_λ of Z described in §4.1 corresponds to the component of Λ^s labelled by the bipartition (λ, \emptyset) . We will write $\Lambda_\lambda^s = \Lambda^s \cap \Lambda_{\lambda, \emptyset}$ for this component, and $\mathcal{M}_\lambda^{\text{sn}}$ for the corresponding components of \mathcal{M}^{sn} .

4.3. We now define a decomposition of Λ^s into locally closed pieces. It will evidently be G -stable, and hence induce a decomposition of Z . In fact for convenience in this subsection we will work mostly with \mathcal{M}^{sn} . Recall that if $(X, Y, i, j) \in \mathcal{M}^s$, then we have $j = 0$, so that X, Y commute with each other. We will thus prefer to write (X, Y, i) for a point in \mathcal{M}^{sn} rather than $(X, Y, i, 0)$.

Definition 4.3.1. Let $(X, Y, i) \in \mathcal{M}^{\text{sn}}$. Let $X = z + x$ be the decomposition of X into its semisimple and nilpotent parts respectively, and let $V = \bigoplus_{i=1}^k V_i$ be the decomposition of V into the eigenspaces of z . Since Y commutes with z it preserves each V_i , and its restriction to V_i is a nilpotent endomorphism with Jordan type $\nu^{(i)}$ say. In this way we obtain a multipartition $\underline{\nu} = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(k)})$, and we may assume that $(\dim(V_1), \dots, \dim(V_k))$ form a partition λ of n . Let $\mathcal{M}_{\lambda, \underline{\nu}}^{\text{sn}}$ denote the set of $(X, Y, i, j) \in \Lambda$ which give rise to the pair $(\lambda, \underline{\nu})$ in this way, where if $\lambda_i = \lambda_{i+1}$ then we must identify the parameters $(\lambda, \underline{\nu})$ and $(\lambda, \underline{\nu}')$ where $\underline{\nu}'$ is obtained from $\underline{\nu}$ by exchanging the i -th and $(i+1)$ th partitions. We will also write $\Lambda_{\lambda, \underline{\nu}}^s$ for the corresponding subset of Λ .

We now wish to show that the pieces of our decomposition are smooth and to calculate their dimension. For this we begin with a technical lemma which gives a sort of normal form for elements of \mathcal{M}^{sn} .

Lemma 4.3.1. *Suppose that $(X, Y, v) \in \mathcal{M}^{sn}$, and that X and Y are nilpotent endomorphisms. For $i, j \in \mathbb{Z}_{\geq 0}$, set $e_{i,j} = X^i Y^j(v)$. Then we may find integers $k, s_0, s_2, \dots, s_{k-1}$, so that $\{e_{i,j} : 0 \leq i < k, 0 \leq j < s_i\}$ is a basis of V , and moreover the action of X on V is given by*

$$X(e_{i,j}) = \begin{cases} e_{i+1,j}, & \text{if } i < k-1, j < s_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

while if $\mathcal{F} = (F_i)$ denotes the decreasing filtration of V given by $F_i = \text{im}(X^i)$, then action of Y must satisfy:

$$Y(e_{i,j}) = \begin{cases} e_{i,j+1}, & \text{if } i < k, j < s_i - 1 \\ w_{i+1}, & \text{if } j = s_i - 1. \end{cases}$$

where w_{i+1} is some vector in F_{i+1} .

Proof. Let k be the nilpotence of degree X , so that $X^k = 0$ but $X^{k-1} \neq 0$, and set $F_i = \text{im}(X^i)$ so that $\mathcal{F} = (F_i)_{0 \leq i \leq k-1}$ is a k -step (decreasing) flag in V . Now since Y commutes with X , and hence with every power of X we immediately see that Y preserves the flag \mathcal{F} , and clearly $X(F_i) = F_{i-1}$. Let $s_i = \dim(F_i/F_{i+1})$ for $0 \leq i < k$, clearly $\mu = (s_0, s_1, \dots, s_{k-1})$ is a partition of n (the dual partition to the one given by the Jordan type of X).

Now the vector $X^{k-1}(v)$ is clearly cyclic for the action of $\mathbb{C}[X, Y]$ restricted to F_{k-1} , but since X acts by zero on F_{k-1} , it follows that $X^{k-1}(v)$ is cyclic for the action of Y on F_{k-1} , and hence $\{e_{k-1,j} : 0 \leq j < s_{k-1}\}$ is a basis for F_{k-1} , and our description of the action of X, Y on this part of the basis is established. But now considering V/F_{k-1} and using induction on k completes the proof. \square

Remark 4.3.1. The $\text{GL}(V)$ orbit of X is of course just the Richardson orbit attached to the parabolic determined by the flag \mathcal{F} .

Lemma 4.3.2. *Suppose that $(X, Y, v) \in \mathcal{M}^{sn}$ and X, Y are nilpotent. Then given $a \geq b \in \mathbb{Z}$, we may find a homomorphism $\rho: \mathbb{C}^\times \rightarrow \text{GL}(V)$ such that*

- (1) $\text{Ad}(\rho(t))(X) = t^a X$,
- (2) $\text{Ad}(\rho(t))(Y) = t^b Y_0 + O(t^a)$,
- (3) $\rho(t)(v) = v$.

Moreover, the triple (X, Y_0, v) lies in \mathcal{M}^{sn} .

Proof. We use the basis constructed in the previous Lemma. For $t \in \mathbb{C}^\times$, let $\rho(t)$ act on V by

$$\rho(t)(e_{i,j}) = t^{-aj-bi} e_{i,j}, \quad (0 \leq i \leq k-1, 0 \leq j \leq s_i).$$

Since $v = e_{0,0}$, condition (3) is immediate, and similarly condition (2) follows readily. Moreover to check condition (1), we simply define Y_0 by

$$Y_0(e_{i,j}) = \begin{cases} e_{i,j+1}, & \text{if } j < s_i; \\ 0 & \text{if } j = s_i. \end{cases}$$

The stability of the triple (X, Y_0, v) is then clear. \square

With these lemmas in hand, we can now check that the pieces of our decomposition are smooth.

Proposition 4.3.1. *Each piece $\mathcal{M}_{\lambda, \underline{\nu}}^{\text{sn}}$ is a connected smooth locally closed subvariety of Λ^s , and if $Z_{\lambda, \underline{\nu}}$ is its image under the quotient map to $Z \subset \text{Hilb}^n(\mathbb{C}^2)$, then $Z_{\lambda, \underline{\nu}}$ is an affine space bundle over $S_\lambda^n(\Sigma) \subset S^n(\mathbb{C}^2)$ via the Hilbert-Chow morphism. Moreover the dimension of the stratum $Z_{\lambda, \underline{\nu}}$ is given by*

$$n - \sum_{i=1}^k (\ell(\nu^{(i)}) - 1).$$

Proof. Since the strata $S_\lambda^n(\Sigma)$ are smooth locally closed subvarieties of \mathbb{C}^n/S_n the last part of the Proposition implies the other claims. Thus it is enough to show that $Z_{\lambda, \underline{\nu}}$ is an affine space bundle over $S_\lambda^n(\Sigma)$.

For this we first examine the case where $\lambda = (n)$, so that $S_\lambda^n(\Sigma)$ is just a point. The component $Z_{(n)}$ is the “punctual” Hilbert scheme, that is the moduli space of codimension n ideals supported at $0 \in \mathbb{C}^2$, and in $T^*\mathfrak{G}^\circ$ it corresponds to the set

$$\mathcal{M}_{(n)}^{\text{sn}} = \{(X, Y, v) \in T^*\mathfrak{G}^\circ : [X, Y] = 0, \mathbb{C}[X, Y]v = \mathbb{C}^n, X, Y \text{ nilpotent}\}$$

For a point $(X, Y, v) \in \mathcal{M}^s$ we will write $[X, Y, v]$ for the corresponding point in the quotient $\text{Hilb}^n(T^*\Sigma)$. We claim that fixing the Jordan type of X describes a locally closed affine space in $Z_{(n)}$. To see this we use a torus action (for more details on the facts used here see for example [Na, Chapter 5]). Let $T^2 = (\mathbb{C}^\times)^2$ acts on $\mathcal{M}_{(n)}^{\text{sn}}$ by

$$(t_1, t_2)(X, Y, v) = (t_1 X, t_2 Y, v).$$

This action descends to the natural action of T^2 on $\text{Hilb}^n(T^*\Sigma)$ induced from the action of T^2 on $T^*\Sigma \cong \mathbb{C}^2$.

Recall the classical Bialynicki-Birula result which shows that a smooth projective variety with a \mathbb{G}_m -action has a natural decomposition into pieces which are affine bundles over the components of the fixed-point locus of that action. Now we cannot apply the Bialynicki-Birula result directly to $Z_{(n)}$, since it is not smooth. However the T^2 action on $Z_{(n)}$ extends to the whole Hilbert scheme $\text{Hilb}^n(\mathbb{P}^2)$ of n points in the projective plane, which is smooth and projective, and hence the Bialynicki-Birula result may be applied to it. In fact, the action has finitely many fixed points, so that the pieces are affine spaces: if $x \in \text{Hilb}^n(\mathbb{P}^2)$ then the piece of the decomposition attached to x is given by:

$$P_x = \{y \in \text{Hilb}^n(\mathbb{P}^2) : \lim_{t \rightarrow \infty} \nu(t) \cdot y = x\}.$$

Choosing the \mathbb{G}_m -action to be given by a generic subgroup $\nu: \mathbb{G}_m \rightarrow T^2$ where $\nu(t) = (t^a, t^b)$ with $a > b > 0$, we see by considering the action of \mathbb{G}_m on the symmetric product $S^n(\mathbb{C}^2)$ (i.e. on the spectra of matrices X and Y) that no point x outside $Z_{(n)}$ can have $\lim_{t \rightarrow \infty} \nu(t) \cdot x$ lying inside $Z_{(n)}$. Thus, since $Z_{(n)}$ is certainly T^2 -invariant, it is a union of pieces of the decomposition of $\text{Hilb}^n(\mathbb{P}^2)$. (See for example [ES, §1] for the statement of the Bialynicki-Birula result and [ES, §2] for a similar use of the theorem for subvarieties of $\text{Hilb}^n(\mathbb{P}^2)$ which are preserved by the torus action).

It is known (see [Na, §5.2] for a detailed discussion) that the T^2 -action (on $\text{Hilb}^n(\mathbb{A}^2)$) has finitely many fixed points corresponding to monomial ideals of codimension n all of which lie in $Z_{(n)}$, thus the pieces in the decomposition of $Z_{(n)}$

are naturally labelled by partitions of n . If x_λ is the monomial ideal corresponding to the partition λ we will write P_λ for the corresponding piece. It is known that P_λ is an affine space of dimension $n - \ell(\lambda)$. Letting $p: \mathcal{M}^s \rightarrow \text{Hilb}^n(\mathbb{C}^2)$ denote the quotient map, we claim that $p^{-1}(P_\lambda) = \mathcal{M}_{(n),\lambda}^{\text{sn}}$.

To show this we use a one-parameter subgroup in $\text{GL}(V)$ as in Lemma 4.3.2. Since the T^2 action commutes with the G -action we have

$$\begin{aligned} \nu(t)[X, Y, \ell] &= [t^a X, t^b Y, \ell] \\ &= [(\rho(t^{-1})(t^a X), \rho(t^{-1})(t^b Y), \rho(t^{-1})\ell)] \\ &= [X, Y_0 + Y_t, \ell], \end{aligned}$$

where $Y_t \in t^{-1}\text{End}(V)[t^{-1}]$. Thus since $\lim_{t \rightarrow \infty} (X, Y_0 + Y_t, v) = (X, Y_0, v)$, which again by Lemma 4.3.2 lies in \mathcal{M}^{sn} , we see that the limit in $Z_{(n)}$ preserves the Jordan type of X . Since this also characterizes the fixed points of the \mathbb{C}^\times -action on $Z_{(n)}$ it follows immediately that $p^{-1}(P_\lambda) = \mathcal{M}_{(n),\lambda}^{\text{sn}}$ as claimed.

In the general case, if $x \in S_\lambda^n(\Sigma)$ then $\pi^{-1}(x)$ is a product of punctual Hilbert schemes, indeed we have

$$\pi^{-1}(x) \cong Z_{(\lambda_1)} \times Z_{(\lambda_2)} \times \dots \times Z_{(\lambda_k)},$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, and the isomorphism is given by taking, for $[X, Y, v] \in \pi^{-1}(x)$, the components of the nilpotent parts³ of X and Y in the eigenspaces of the semisimple part of X , along with the projection of the vector v to that eigenspace. It follows immediately that the intersection of the $Z_{\lambda, \underline{\nu}}$ with $\pi^{-1}(x)$ is a product of affine spaces, and hence itself an affine space. Moreover, the dimension of the affine space is clearly

$$\sum_{i=1}^k (\lambda_i - \ell(\nu^{(i)})) = n - \sum_{i=1}^k \ell(\nu^{(i)}).$$

But since $S_\lambda^n(\Sigma)$ clearly has dimension $k = \ell(\lambda)$, it follows that

$$\dim(Z_{\lambda, \underline{\nu}}) = n - \sum_{i=1}^k (\ell(\nu^{(i)}) - 1).$$

as claimed. \square

Remark 4.3.2. Notice that it follows from the dimension formula that the pieces of the decomposition of maximal dimension are the pieces $Z_{\lambda, \underline{\nu}}$ where

$$\underline{\nu} = ((\lambda_1), (\lambda_2), \dots, (\lambda_k)).$$

Notice moreover that the pieces of our decomposition which have dimension $n - 1$ are those for which exactly one partition $\nu^{(i)}$ has two parts. From this it follows that such a piece lies in exactly two components of Z , those being Z_λ and $Z_{\lambda'}$ where λ' is obtained from λ by replacing λ_i by the two parts of $\nu^{(i)}$. Thus the decomposition of Z also allows us to describe which components of Z intersect in codimension 1.

For example, the components which intersect the punctual Hilbert scheme $Z_{(n)}$ in codimension 1 are those of the form $Z_{(k, n-k)}$, while the only component intersecting $Z_{(1^n)}$ in codimension 1 is $Z_{(2, 1^{n-2})}$.

³Of course Y is nilpotent, so is equal to its nilpotent part.

4.4. We now use our decomposition of \mathcal{M}^{sn} to find the smooth locus of \mathcal{M}^{sn} . We claim that its components are exactly the maximal dimensional pieces of our decomposition. Thus we set $\tilde{\mathcal{M}}_\lambda^{\text{sn}} = \mathcal{M}_{\lambda, \underline{\nu}}^{\text{sn}}$ where $\nu = ((\lambda_1), (\lambda_2), \dots, (\lambda_k))$, that is, each partition in $(\underline{\nu})$ has a single part. By the dimension formula in Proposition 4.3.1 each $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ has dimension equal to $\dim(\Lambda^s)$. Let \tilde{Z}_λ and $\tilde{\Lambda}_\lambda$ denote the corresponding pieces of Z and Λ^s respectively.

Lemma 4.4.1. *Let $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ be as above. Then $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ is an open dense subvariety of $\mathcal{M}_\lambda^{\text{sn}}$ consisting of smooth points of \mathcal{M}^{sn} .*

Proof. Since $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ is a piece of our decomposition, we have already established that it is smooth. To conclude that $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ consists of smooth points of \mathcal{M}^{sn} it remains to check that $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ does not intersect any other component of \mathcal{M}^{sn} . For this suppose that $(X_0, Y_0, v_0) \in \tilde{\mathcal{M}}_\lambda^{\text{sn}}$ lies in some other component of \mathcal{M}^{sn} , say $\mathcal{M}_\mu^{\text{sn}}$. Then since $\tilde{\mathcal{M}}_\mu^{\text{sn}}$ is dense in $\mathcal{M}_\mu^{\text{sn}}$ we see that (X_0, Y_0, v_0) must lie in the closure of $\tilde{\mathcal{M}}_\mu^{\text{sn}}$. Thus suppose that $(X_i, Y_i, v_i)_{i \geq 1} \in \tilde{\mathcal{M}}_\mu^{\text{sn}}$ converges⁴ to (X_0, Y_0, v_0) . But now the Jordan type of each Y_i is given by μ , and since the (Y_i) converges to Y_0 we must have $\mu \geq \lambda$ in the dominance ordering on partitions, and hence certainly we must have $\ell(\mu) \leq \ell(\lambda)$.

On the other hand, letting $s: \mathcal{M}^{\text{sn}} \rightarrow \mathfrak{h}/S_n$ be the map given by taking the eigenvalues of X . Since the sequence $s(X_i, Y_i, v_i)$ consists of $\ell(\mu)$ distinct points with multiplicities μ_i , $(1 \leq i \leq \ell(\mu))$, it follows that either X_0 has at most $\ell(\mu)$ distinct eigenvalues and so $\ell(\mu) \geq \ell(\lambda)$. Hence we see that we must have $\ell(\mu) = \ell(\lambda)$, and then moreover the eigenvalue multiplicities of X_0 must be equal to those of the X_i 's (as there can be no "collision" of eigenvalues), whence we have $\lambda = \mu$ as desired. \square

Remark 4.4.1. It is easy to see that every other piece in our decomposition lies in more than one component of Z , and hence the union of our subsets \tilde{Z}_λ in fact yields the entire smooth locus of Z .

Example 4.4.1. If $\lambda = (n)$, then Y has a single Jordan block, and the space $\tilde{\Lambda}_\lambda$ is just an affine space bundle over \mathcal{O}_Y with fibres of dimension $2n - 1$. The corresponding subset in $Z_{(n)}$ is an affine space of dimension $n - 1$, it is the open cell in an affine paving of $Z_{(n)}$.

We now wish to compute the fundamental group of the smooth loci $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$. For this we make the following definition.

Definition 4.4.2. Let λ be a partition of n . We may write λ uniquely as $(i_1^{c_1}, i_2^{c_2}, \dots, i_r^{c_r})$ where $1 \leq i_1 < i_2 < \dots < i_r \leq n$. Let $k = \ell(\lambda) = \sum_{i=1}^r c_i$. Let \mathcal{B}_λ be the fundamental group of the configuration space of k labelled points in \mathbb{C} where the labels lie in $\{i_1, i_2, \dots, i_r\}$ and there are c_j points with label i_j . In our earlier notation this is just the space $S_\lambda^n(\mathbb{C})$. Let \mathcal{P}_λ be the subgroup of \mathcal{B}_λ corresponding to the cover of the configuration space where all k points are distinct, i.e. the pure braid group on k strands, and let Σ_λ be the quotient $\mathcal{B}_\lambda / \mathcal{P}_\lambda$ so that $\Sigma_\lambda \cong S_{a_1} \times S_{a_2} \times \dots \times S_{a_k}$, where S_a denotes the symmetric group on a letters.

⁴For this argument, we use the complex topology on our varieties, not the Zariski topology.

Thus for example if $\lambda = (1^n)$ this group is the standard braid group on n strands. Note that \mathcal{B}_λ is a subgroup of the braid group on k strands. It contains the parabolic braid subgroup given by the partition associated to the composition (c_1, c_2, \dots, c_r) , but is strictly bigger – for example it contains the isotopies which rotate points labeled i and j a full 2π about each other.

The fundamental group of $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ is closely related to \mathcal{B}_λ as we now show. For $\mathbf{a} = (a_1, a_2, \dots, a_{r-1}) \in \mathbb{C}^{r-1}$, let

$$J_{\mathbf{a}}(t) = \begin{pmatrix} t & a_1 & a_2 & \dots & a_{r-2} & a_{r-1} \\ 0 & t & a_1 & & & a_{r-2} \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_1 & a_2 \\ 0 & 0 & 0 & 0 & t & a_1 \\ 0 & 0 & 0 & 0 & 0 & t \end{pmatrix} \in \text{Mat}_r(\mathbb{C}).$$

Let $\mu = \{\mu_1, \dots, \mu_k\} \in S_\lambda^n(\mathbb{C})$ with labels (*i.e.* multiplicities) given by the parts of λ written in increasing order: $d_1 \leq d_2 \leq \dots \leq d_k$ so that $d_j \in \{i_1, i_2, \dots, i_r\}$ for each j , ($1 \leq j \leq k$). Then if $\mathbf{a}^j \in \mathbb{C}^{d_j-1}$ for $1 \leq j \leq k$, set $X_\lambda(\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^k)$ to be the block diagonal matrix:

$$\begin{pmatrix} J_{\mathbf{a}^1}(\mu_1) & 0 & 0 & 0 \\ 0 & J_{\mathbf{a}^2}(\mu_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{\mathbf{a}^k}(\mu_k) \end{pmatrix}$$

and similarly set Y_λ to be the block diagonal nilpotent matrix with Jordan blocks of size d_1, d_2, \dots, d_k . Finally, set $v_\lambda \in \mathbb{C}^n$ to be the vector with $v_j = 1$ if $j \in \{\sum_{s=1}^t d_s : 1 \leq t \leq k\}$ and $v_j = 0$ otherwise. It is then easy to check that $(X_\lambda(\mu, \mathbf{a}^1, \dots, \mathbf{a}^k), Y_\lambda, v_\lambda) \in \tilde{\mathcal{M}}_\lambda^{\text{sn}}$, and moreover if $(X_\lambda(\mu, \mathbf{a}^1, \dots, \mathbf{a}^k), Y_\lambda, v_\lambda)$ and $(X_\lambda(\mu', \mathbf{b}^1, \dots, \mathbf{b}^k), Y_\lambda, v_\lambda)$ are two such points then they lie in the same $\text{GL}(V)$ -orbit if and only if they are equal up to permutation of the Jordan blocks of the X s – *i.e.* conjugate by an element Σ_λ viewed as the subgroup of the group of permutation matrices which interchanges our diagonal blocks of the same size (respecting the order within each block). Thus if we set \mathcal{S}_λ to be the subspace of $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ consisting of points of the form $(X(\mu, \mathbf{a}^1, \dots, \mathbf{a}^k), Y_\lambda, v_\lambda)$, then we have a natural map

$$p_\lambda: \text{GL}(V) \times \mathcal{S}_\lambda \rightarrow \tilde{\mathcal{M}}_\lambda^{\text{sn}},$$

given by the $\text{GL}(V)$ -action, and this map is readily seen to be a Σ_λ -covering, where Σ_λ acts on $\text{GL}(V)$ on the right (embedded as a subgroup of permutation matrices) and on \mathcal{S}_λ by the restriction of the $\text{GL}(V)$ -action on \mathcal{M}^{sn} .

Proposition 4.4.1. *Let λ be a partition of n . Then we have exact sequences:*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\tilde{\mathcal{M}}_\lambda^{\text{sn}}) \longrightarrow \mathcal{B}_\lambda \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z} \times \mathcal{P}_\lambda \longrightarrow \pi_1(\tilde{\mathcal{M}}_\lambda^{\text{sn}}) \longrightarrow \Sigma_\lambda \longrightarrow 1$$

Moreover, \mathcal{P}_λ is normal in $\pi_1(\tilde{\mathcal{M}}_\lambda^{\text{sn}})$, the subgroup \mathbb{Z} is central, and $\pi_1(\tilde{Z}_\lambda) \cong \mathcal{B}_\lambda$.

Proof. First notice that the variety \tilde{Z}_λ is homotopy equivalent to $S_\lambda^n(\Sigma)$ since it is an affine space bundle over it, thus clearly $\pi_1(\tilde{Z}_\lambda) \cong \mathcal{B}_\lambda$. Now since the configuration space of n distinct points in the complex plane is known to be a $K(\pi, 1)$

(see for example [D]), it follows that $S_\lambda^n(\Sigma)$ is a $K(\pi, 1)$. The long exact sequence of a fibration then immediately shows that the fundamental group of $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ is an extension of \mathcal{B}_λ by $\pi_1(\text{GL}(V)) \cong \mathbb{Z}$. This yields the first of our exact sequences.

Now consider the covering p_λ . Since the total space is a product, the theory of covering spaces yields the second exact sequence. Moreover, the regular covering of $\tilde{\mathcal{M}}_\lambda^{\text{sn}}$ given by taking the universal cover of $\text{GL}(V)$ over the total space of p_λ shows that \mathcal{P}_λ is a normal subgroup of $\pi_1(\tilde{\mathcal{M}}_\lambda^{\text{sn}})$. Thus since \mathbb{Z} is clearly centralised by \mathcal{P}_λ it suffices to show that \mathbb{Z} is central in the quotient by \mathcal{P}_λ , which is isomorphic to the fundamental group of $\text{GL}(V)/\Sigma_\lambda$. But this is equivalent to showing that the action of Σ_λ on $\pi_1(\text{GL}(V))$ is trivial which is immediate, because the action of Σ_λ is the restriction of the action of the connected group $\text{GL}(V)$. \square

Remark 4.4.2. Note that one can also describe the fundamental group of $\text{GL}(V)/\Sigma_\lambda$ explicitly, for example by considering it as a subgroup of the universal cover \tilde{G} of $\text{GL}(V)$. Indeed $\tilde{G} = \{(g, t) \in \text{GL}(V) \times \mathbb{C} : \det(g) = \exp(2\pi it)\}$, and we may realise

$$\pi_1(\text{GL}(V)/\Sigma_\lambda) = \{(w, n) \in \Sigma_\lambda \times (1/2)\mathbb{Z} : \text{sgn}(w) = \exp(2\pi in)\}$$

where sgn is the usual sign function on S_n restricted to Σ_λ . Here the subgroup $\mathbb{Z} \cong \pi_1(\text{GL}(V))$ is realized as $\{(1, n) : n \in \mathbb{Z}\}$. For use in the next section, we also note here that, if we fix a generator γ of $\pi_1(\text{GL}(V))$ and take any $q \in \mathbb{C}^\times$, say $q = \exp(2\pi i\alpha)$, then $\pi_1(\tilde{\mathcal{M}}_\lambda^{\text{sn}})$ has a one-dimensional representation L_q on which γ acts by q . Indeed the quotient $\pi_1(\text{GL}(V)/\Sigma_\lambda)$ obviously has such a representation, by taking the restriction of the representation $\Sigma_\lambda \times (1/2)\mathbb{Z}$ which sends $(w, n) \mapsto \exp(2\pi in\alpha)$. It is easy to check that in the case $\lambda = (1^n)$ the representations L_q are given by the monodromy of the \mathcal{D} -module s^c , where s is as in §3.3. We will use this fact in comparing our microlocal KZ -functors to the original KZ functor.

5. MICROLOCAL KZ FUNCTORS

5.1. Let $\mathcal{O}_c^{\text{sph}}$ be the category of representations of the spherical rational Cherednik algebra corresponding to the category \mathcal{O}_c of modules for the rational Cherednik algebra, in other word, $\mathcal{O}_c^{\text{sph}}$ is the category of $e\mathcal{H}_c e$ -modules on which $\mathbb{C}[\mathfrak{h}^*]_+^W$, the augmentation ideal of $\mathbb{C}[\mathfrak{h}^*]^W$, acts locally nilpotently. In this section we use our analysis of the variety Λ to study $\mathcal{O}_c^{\text{sph}}$.

It is well known that projective spaces are \mathcal{D} -affine. More precisely, if we assume⁵ that $c \notin \frac{1}{n}\mathbb{Z}_{<0}$ then the localization functor gives an equivalence between $\mathcal{D}_{\mathfrak{X},c}$ -modules and $\mathcal{D}_c(\mathfrak{X})$ -modules. Next recall the adjoint Loc of the quantum Hamiltonian reduction functor, (in [GG] the corresponding functor is denoted ${}^\top \mathbb{H}$). Let \mathcal{L}_c be the $\mathcal{D}_{\mathfrak{X},c}$ -module whose global sections are:

$$\mathcal{D}_c(\mathfrak{X})/\mathcal{D}_c(\mathfrak{X}) \cdot \mathfrak{g}_c,$$

where \mathfrak{g}_c is the image of \mathfrak{sl}_n under the twisted quantized moment map τ_c associated to the action of $\text{PGL}(V)$ on \mathfrak{X} via the isomorphism $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n \rightarrow \text{pgl}_n$. The twisted Harish-Chandra homomorphism Ψ_c defined in Section 3 yields an isomorphism between $e\mathcal{H}_c e$ and $\text{End}(\mathcal{L}_c)^{\text{op}}$, so that given a $e\mathcal{H}_c e$ -module M we may

⁵The $\frac{1}{n}$ factor comes from the fact that we use $\text{tr}_{|\text{Lie}(\mathcal{Z})}$ as a basis vector for the dual space of $\text{Lie}(\mathcal{Z})$.

associate to it a $\mathcal{D}_{\mathfrak{X},c}$ -module

$$\mathrm{Loc}(M) = \mathcal{L}_c \otimes_{e\mathcal{H}_ce} M.$$

This is the left adjoint to the functor $\mathbb{H}: \mathcal{C}_c \rightarrow \mathcal{O}_c^{\mathrm{sph}}$ which sends $\mathcal{F} \mapsto \Gamma(\mathcal{F})^{\mathrm{SL}(V)}$, and it is known that the adjunction morphism $\mathbb{H} \circ \mathrm{Loc}(M) \rightarrow M$ is an isomorphism for all $M \in \mathrm{ob}(\mathcal{O}_c^{\mathrm{sph}})$. Recall that since $\mathrm{SL}(V)$ commutes with the action of $\mathcal{Z} \subset \mathrm{GL}(V)$ on \mathfrak{G}° , we may consider the category of $\mathrm{SL}(V)$ -equivariant $\mathcal{D}_{\mathfrak{X},c}$ -modules (see Appendix 9.1 for details and references). In [GG], Gan and Ginzburg establish the following result: let \mathcal{C}_c be the full subcategory of the category of $(\mathcal{D}_{\mathfrak{X},c}, \mathrm{SL}(V))$ -modules whose objects have their characteristic cycle contained in Λ .

Theorem 5.1.1. [GG, §7]. *Suppose that $c \notin \frac{1}{n}\mathbb{Z}_{<0}$. Then the functor Loc yields an equivalence of categories between $\mathcal{O}_c^{\mathrm{sph}}$ and $\mathcal{C}_c/\ker(\mathbb{H})$.*

Remark 5.1.1. Here $\ker(\mathbb{H})$ is the subcategory of objects in \mathcal{C}_c which are annihilated by \mathbb{H} . Since $\mathrm{SL}(V)$ is reductive, the functor of Hamiltonian reduction \mathbb{H} is exact, so that $\ker(\mathbb{H})$ is a Serre subcategory. Kashiwara and Rouquier use the essentially same procedure, but then descend to the Hilbert scheme.

5.2. We now wish to construct our microlocal analogues of the KZ -functors. For this we need the language of microdifferential operators. The most “algebraic” approach is to work with the ring $\widehat{\mathcal{E}}_X$ of formal microdifferential operators (see [K6, Chapter 7] where it is constructed using the calculus of total symbols) however there are a number of variants: the ring \mathcal{E}_X of microdifferential operators with growth conditions, the sheaf \mathcal{E}_X^∞ of infinite order operators, and the sheaf $\mathcal{E}_X^{\mathbb{R}}$, see for example [K4] for a survey.

We briefly recall some basic properties (in the formal setting). Given X a complex manifold, $\widehat{\mathcal{E}}_X$ is a \mathbb{Z} -filtered sheaf of rings on T^*X . If $\pi: T^*X \rightarrow X$ is the bundle map then $\widehat{\mathcal{E}}_X$ contains the sheaf $\pi^{-1}\mathcal{D}_X$ as a subring, and the quotients $\widehat{\mathcal{E}}_X(m)/\widehat{\mathcal{E}}_X(m-1) \cong \mathcal{O}_{T^*X}(m)$ where $\mathcal{O}_{T^*X}(m)$ denotes the sheaf of holomorphic functions on T^*X homogeneous of degree m along the fibres of T^*X . It is also known that $\widehat{\mathcal{E}}_X$ is flat over $\pi^{-1}\mathcal{D}_X$, and one can describe the characteristic cycle of a \mathcal{D}_X -module in terms of the associated $\widehat{\mathcal{E}}_X$ -module: indeed for a coherent \mathcal{D}_X -module \mathcal{F} we have:

$$(5.1) \quad \mathrm{Supp}(\widehat{\mathcal{E}}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{F}) = \mathrm{SS}(\mathcal{F}),$$

where SS denotes singular support (the multiplicities given by the characteristic cycle can also be recovered as discussed below). Note that when working with coherent \mathcal{D}_X -modules, we often consider a good filtration. In the context of $\widehat{\mathcal{E}}_X$ -modules this corresponds to taking an $\widehat{\mathcal{E}}_X(0)$ -lattice.

Suppose that \mathcal{F} is a coherent \mathcal{D}_X -module, and U is an open subset of X . It is well-known that if $\mathrm{CC}(\mathcal{F})|_{T^*U} \subseteq U$ then $\mathcal{F}|_U$ is in fact \mathcal{O}_U -coherent, and hence a vector bundle with connection, or local system, on U . The theory of regular holonomic $\widehat{\mathcal{E}}_X$ -modules shows that this local constancy has a natural generalization to other Lagrangian submanifolds of T^*X . Let \mathring{T}^*X denote the complement to the zero section in T^*X . If Ω is an open subset of T^*X , then we define a functor μ_Ω from \mathcal{D}_X -modules to $\widehat{\mathcal{E}}_X|_\Omega$ -modules by setting $\mu_\Omega(\mathcal{F}) = (\widehat{\mathcal{E}}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}(\mathcal{F}))|_\Omega$.

Theorem 5.2.1. *Let Ω be an open subset of \mathring{T}^*X and let Λ be a homogeneous smooth Lagrangian submanifold in Ω such that the projection from Λ to X has constant rank. Then there is an exact functor Φ_Λ from the category of holonomic $\widehat{\mathcal{E}}_X$ -modules \mathcal{N} with $SS(\mathcal{N}) \cap \Omega \subset \Lambda$ to the category of local systems on Λ .*

Proof. This can be seen topologically or analytically. One approach works with the ring $\mathcal{E}_X^\mathbb{R} = \mathcal{H}^n(\mu_\Delta(\mathcal{O}_{X \times X}^{(0,n)}))$, where μ_Δ is the microlocalization functor, $n = \dim(X)$, and $\mathcal{O}_{X \times X}^{(0,n)}$ is the sheaf of holomorphic forms on $X \times X$ which are n -forms with respect to the second variable. This is a sheaf of rings which contains \mathcal{E}_X and is faithfully flat over it.

Now by the assumption on the projection from Λ to X , we may write $\Lambda = T_Z^*X \cap \Omega$ for some submanifold Z of X . Then we have the simple holonomic system $\mathcal{C}_{Z|X} = \mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{B}_{Z|X}$ along Λ . Now for any holonomic \mathcal{E}_X -module \mathcal{N} by⁶ [KK1, Theorem 1.3.1(i)] we have $\text{Ext}_{\mathcal{E}_X}^j(\mathcal{C}_{Z|X}, \mathcal{N}^\mathbb{R}) = 0$ for $j \neq 0$ and moreover $\text{Hom}_{\mathcal{E}_X}(\mathcal{C}_{Z|X}, \mathcal{N}^\mathbb{R})$ is a locally constant sheaf on Λ . Thus the map

$$\mathcal{M} \mapsto \text{Hom}_{\mathcal{E}_X}(\mathcal{C}_{Z|X}, \mathcal{N}^\mathbb{R}),$$

where $\mathcal{N} = (\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{M})|_\Omega$, yields an exact functor to local systems.

Alternatively, via the Riemann-Hilbert correspondence, we may work topologically with the perverse sheaf P determined by \mathcal{M} . In this situation the functor to local systems is the vanishing cycles functor, and its construction is described in [MV1, §5]. A more detailed discussion, giving a number of different approaches in the topological setting, is given in [GMV, §4]. \square

Remark 5.2.1. Note that [KK1, Theorem 1.3.1(i)] also states that there is a canonical isomorphism:

$$\mathcal{C}_{Z|X}^\mathbb{R} \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{E}_X}(\mathcal{C}_{Z|X}, \mathcal{N}^\mathbb{R}) \rightarrow \mathcal{N}^\mathbb{R}.$$

where $\mathcal{N}^\mathbb{R}$ denotes the extension of \mathcal{N} to $\mathcal{E}_X^\mathbb{R}$. If $\Lambda = \gamma^{-1}\gamma(\Lambda)$ where $\gamma: \mathring{T}^*X \rightarrow \mathbb{P}^*X$ is the map to the projectivised cotangent bundle, then by [KK1, Theorem 1.3.1 (ii)] the sheaf \mathcal{M}^∞ (the extension of \mathcal{M} to the ring \mathcal{E}_X^∞ of infinite order microdifferential operators) is completely determined by $\mathcal{M}^\mathbb{R}$, and if \mathcal{M} has regular singularities, then by [KK1, Theorem 5.2.1] it is determined by \mathcal{M}^∞ . Thus in this case there is in fact an equivalence between regular singularities $\mathcal{E}_{X|\Omega}$ -modules supported on Λ and local systems on Λ . In general however, regular singularities \mathcal{E}_X -modules supported on a smooth homogeneous Lagrangian are equivalent to twisted local systems on Λ (see [K4, §10] for more details).

Finally, using for example the explicit description of the functor Φ_Λ as homomorphisms from $\mathcal{C}_{Z|X}$ we see that if our holonomic modules are (H, λ) -equivariant for some group H , then Φ_Λ restricts to a functor taking values in the category of (H, λ) -twisted local systems on Λ .

Remark 5.2.2. The condition that U is an open in \mathring{T}^*X is not particularly restrictive: if we start with a $\widehat{\mathcal{E}}_X$ -module \mathcal{F} we may replace X by $Z = X \times \mathbb{C}$ and \mathcal{F} by $\mathcal{F} \boxtimes \delta_0$ where δ_0 denotes the module generated by the δ -function at $0 \in \mathbb{C}$. Since the conormal bundle of $X \times \{0\}$ in Z is now $X \times T_0^*\mathbb{C}$ we have effectively moved the support of \mathcal{F} off the zero section.

⁶The proof of Theorem 1.3.1 in that paper is given in [K2].

5.3. By the discussion in §3.2 we may view the category \mathcal{C}_c of §5.1 as the abelian subcategory of $(\mathrm{GL}(V), c.\mathrm{tr})$ -equivariant module on \mathfrak{G}° whose singular support is contained in $\mathcal{M}^{\mathrm{sn}}$ (the preimage of Λ). Given a component $\mathcal{M}_\lambda^{\mathrm{sn}}$ of $\mathcal{M}^{\mathrm{sn}}$ we may choose an open set Ω_λ in $T^*\mathfrak{G}^\circ$ such that $\Omega_\lambda \cap \mathcal{M}^{\mathrm{sn}} = \tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$. From our explicit description of $\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$ in §4.4 it immediately follows that the projection from $\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$ to \mathfrak{G}° has constant rank (indeed the fibre of the projection from $\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$ over a point $(X, v) \in \mathfrak{G}^\circ$ is just the space of nilpotent matrices which are regular in each generalized eigenspace of X , thus in fact the restriction of the projection is a fibre bundle with smooth fibres). Thus by composing the functor of Theorem 5.2.1 with the functor μ_{Ω_λ} , we obtain an exact functor

$$\mathcal{KZ}_\lambda: \mathcal{C}_c \rightarrow \mathcal{LS}(\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}).$$

where $\mathcal{LS}(\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}})$ denotes the category of local systems on $\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$. Note that the Lagrangian $\tilde{\mathcal{M}}_{(1^n)}^{\mathrm{sn}}$ is a subset of \mathfrak{G}° , so that in that case we only need the classical equivalence between vector bundles with flat connection and local systems.

Definition 5.3.1. By choosing a base point in $\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}$ we may think of \mathcal{KZ}_λ as a functor to the category $\mathrm{Rep}(\pi_1(\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}}))$ of finite-dimensional representations of $\pi_1(\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}})$. Moreover, since we are working with $(\mathrm{GL}(V), c.\mathrm{tr})$ -equivariant modules \mathcal{F} , the local system $\mathcal{KZ}_\lambda(\mathcal{F})$ must also be. Now by Proposition 4.4.1 we see that $\pi_1(\tilde{\mathcal{M}}_\lambda^{\mathrm{sn}})$ is an extension of \mathcal{B}_λ by $\pi_1(\mathrm{GL}(V)) \cong \mathbb{Z}$. The twisted equivariance shows that the subgroup $\mathbb{Z} = \pi_1(\mathrm{GL}(V))$ must act by $q = \exp(2\pi i nc)$. Thus it follows that the tensor product $L_{q^{-1}} \otimes \mathcal{KZ}_\lambda(\mathcal{F})$ descends to a representation of \mathcal{B}_λ , where $L_{q^{-1}}$ is the one-dimensional representation constructed in Remark 4.4.2. Let $KZ_\lambda(\mathcal{F})$ denote this representation of \mathcal{B}_λ , an object of the category of finite-dimensional \mathcal{B}_λ -representations $\mathrm{Rep}(\mathcal{B}_\lambda)$. By composing with the functor Loc we also obtain a functor from $\mathcal{O}_c^{\mathrm{sph}}$ to $\mathrm{Rep}(\mathcal{B}_\lambda)$. Since it should be clear from context, we will abuse notation slightly and again write KZ_λ for this composition of functors⁷.

Remark 5.3.1. Since the representation $L_{q^{-1}}$ is not canonically defined, it is perhaps more natural to have the functors KZ_λ take values in the category of twisted local systems. However, in order to connect more directly with the original KZ functor we prefer here to descend to ordinary local systems.

For use in the next section, we also wish to note the connection between our functors and the characteristic cycles for modules in $\mathcal{O}_c^{\mathrm{sph}}$ defined by Gan-Ginzburg [GG, §7.5] and Gordon-Stafford [GS, §2.7] (which are shown to be equivalent in [GGS, §1.10]). If M is a module in $\mathcal{O}_c^{\mathrm{sph}}$, then the definition in [GG] uses the intersection of $\mathrm{CC}(\mathrm{Loc}(M))$ with the stable locus U , and takes its quotient by $\mathrm{GL}(V)$ to obtain Lagrangian cycle in $\mathrm{Hilb}^n(\mathbb{C}^2)$. By equation (5.1), or more precisely by its refinement which counts multiplicities, this characteristic cycle is obtained by taking the dimension of our local systems $KZ_\lambda(M)$.

5.4. We now relate our functors to the results in [KR]. Recall⁸ that they construct a \mathscr{W} -algebra on $\mathrm{Hilb}^n(\mathbb{C}^2)$ by “symplectic reduction” (see also 8.5). Starting with

⁷Of course one also obtains a functor from \mathcal{O}_c to $\mathrm{Rep}(\mathcal{B}_\lambda)$ by composing with the functor from \mathcal{O}_c to $\mathcal{O}_c^{\mathrm{sph}}$ sending $M \mapsto eM$.

⁸We refer the reader to their paper for details and terminology such as \mathscr{W} -algebras and F -structures, but note we summarise their work using our notation not theirs.

the \mathcal{D} -module \mathcal{L}_c of §5.1, they extend coefficients to obtain a module (also denoted \mathcal{L}_c) for $\mathcal{W}_{T^*\mathfrak{G}}$ the standard \mathcal{W} -algebra on $T^*\mathfrak{G}$, which is $(\mathrm{GL}(v), c.\mathrm{tr})$ -equivariant and is supported on \mathcal{M} . Since $\mathrm{GL}(V)$ acts freely on U the stable locus in $T^*\mathfrak{G}$, once we restrict to U , we may apply the reduction procedure given by Proposition 2.8 of [KR] to obtain a \mathcal{W} -algebra, denoted \mathcal{A}_c , on $\mathcal{M}^s/G \cong \mathrm{Hilb}^n(\mathbb{C}^2)$. Let $\tilde{\mathcal{A}}_c$ denote the \mathcal{W} -algebra obtained from \mathcal{A}_c by extending the coefficients $\mathbb{C}((\hbar))$ by a square root of \hbar . Then there is a natural functor from $e\mathcal{H}_ce$ -modules to $\tilde{\mathcal{A}}_c$ -modules with an F -action given by:

$$\mathrm{Loc}_{\tilde{\mathcal{A}}_c}(M) = \tilde{\mathcal{A}}_c \otimes_{e\mathcal{H}_ce} M.$$

This uses the deformed Harish-Chandra homomorphism to construct a natural action of $e\mathcal{H}_ce$ on $\tilde{\mathcal{A}}_c$ (i.e. the action is induced by the $e\mathcal{H}_ce$ -action on the $\mathcal{D}_{\mathfrak{G}}$ -module \mathcal{L}_c).

By [KR, Proposition 2.8], the construction of the \mathcal{W} -algebra on $\mathrm{Hilb}^n(\mathbb{C}^2)$ has the property that its modules are equivalent to $(G, c.\mathrm{tr})$ -equivariant \mathcal{W}_U -modules with an F -action that are supported on \mathcal{M}^s . Thus comparing the definitions, we see that given a module in category \mathcal{O}_c , the \mathcal{W}_U -module we obtain by pulling back $\mathrm{Loc}_{\tilde{\mathcal{A}}_c}(M)$ to U is exactly $\mathcal{W}_U \otimes_{\hat{\mathcal{E}}_U} \mu_U(\mathrm{Loc}(M))$. Now both Loc and $\mathrm{Loc}_{\tilde{\mathcal{A}}_c}$ have left adjoints \mathbb{H} and $\mathbb{H}_{\tilde{\mathcal{A}}_c}$ given by taking twisted invariant global sections, and moreover, since by [GG, Theorem 1.5] and [KR, Lemma 4.7] there are filtration preserving isomorphisms $\mathrm{End}(\mathcal{L}_c)^{\mathrm{op}} \cong e\mathcal{H}_ce \cong \mathrm{End}_{\mathrm{Mod}_F^{\mathrm{good}}(\tilde{\mathcal{A}}_c)}(\tilde{\mathcal{A}}_c)^{\mathrm{op}}$ it follows we must have $\mathbb{H}_{\tilde{\mathcal{A}}_c}(\mathcal{W}_U \otimes_{\hat{\mathcal{E}}_U} \mathrm{Loc}(M)) = \mathbb{H}(\mathrm{Loc}(M))$.

Now let

$$\mathcal{Y} = \{a/b : 2 \leq b \leq n, a < 0, \mathrm{g.c.d.}(a, b) = 1\}.$$

Then Theorem 4.9 of [KR] shows that if $c \notin \mathcal{Y}$ the functor $\mathbb{H}_{\tilde{\mathcal{A}}_c}$ gives an equivalence between the category of coherent good $\tilde{\mathcal{A}}_c$ -modules with an F -structure and the category of finitely generated $e\mathcal{H}_ce$ -modules, with quasi-inverse $\mathrm{Loc}_{\tilde{\mathcal{A}}_c}$. But now if \mathcal{F} is an object in \mathcal{C}_c , then the natural map $\mathcal{F} \rightarrow \mathrm{Loc}(\mathbb{H}(\mathcal{F}))$ must be an isomorphism over U (since \mathcal{W}_X is faithfully flat over $\hat{\mathcal{E}}_X$). Note that this implies that the functors \mathcal{KZ}_λ which we have defined vanish on $\ker(\mathbb{H})$, and so descend to the quotient category $\mathcal{C}_c/\ker(\mathbb{H})$.

5.5. The case $\lambda = (1^n)$ corresponds to the original KZ -functor of [GGOR], as we now show. Let $\mathfrak{G}^{\mathrm{rss}}$ be the set of pairs $(X, v) \in \mathfrak{G}$ such that X is regular semisimple and v is a cyclic vector for X . This is a smooth locally closed subvariety of \mathfrak{G}° , and by the characteristic cycle bound, if $\mathcal{F} \in \mathcal{C}_c$ then, viewing \mathcal{F} as before as a $(\mathrm{GL}(V), c.\mathrm{tr})$ -equivariant $\mathcal{D}_{\mathfrak{G}^\circ}$ -module, its restriction $\mathcal{F}|_{\mathfrak{G}^{\mathrm{rss}}}$ must be a local system. As in Lemma 4.4.1 we see that that $\pi_1(\mathfrak{G}^{\mathrm{rss}})$ is an extension of \mathcal{B}_n by $\mathbb{Z} \cong \pi_1(\mathrm{GL}(V))$, where \mathcal{B}_n is the braid group on n strands, so that the functor $KZ_{(1^n)}$ given by descending the local system $\mathcal{F}|_{\mathfrak{G}^{\mathrm{rss}}}$ to $\hat{Z}_{(1^n)}$.

Proposition 5.5.1. *The functors KZ and $KZ_{(1^n)}$ are naturally isomorphic.*

Proof. This follows immediately from the construction of the functors. Consider the diagram:

$$\begin{array}{ccccc} \mathfrak{G}^{\mathrm{reg}} & \xrightarrow{\pi} & \mathfrak{h}/W & \xleftarrow{p} & \mathfrak{h} \\ \uparrow j & & \uparrow & & \uparrow i \\ \mathfrak{G}^{\mathrm{rss}} & \xrightarrow{\pi_r} & \mathfrak{h}^{\mathrm{reg}}/W & \xleftarrow{p_r} & \mathfrak{h}^{\mathrm{reg}} \end{array}$$

The construction of the KZ -functor uses the right-hand square in the diagram. Given a module M in category \mathcal{O}_0 , we may localize to $\mathfrak{h}^{\text{reg}}$, and via the Dunkl homomorphism view it as a W -equivariant \mathcal{D} -module on $\mathfrak{h}^{\text{reg}}$ (which we will again write as M). As such there is a \mathcal{D} -module N which is a vector bundle with connection on $\mathfrak{h}^{\text{reg}}$ such that $M|_{\mathfrak{h}^{\text{reg}}} \cong p^*(N)$.

The global sections of N become isomorphic to $eM|_{\mathfrak{h}^{\text{reg}}}$ as an $\mathcal{D}_{\mathfrak{h}^{\text{reg}}}^W$ -module after conjugation by δ : over $\mathfrak{h}^{\text{reg}}$ the map to \mathfrak{h}/W is a Galois covering with group W , so locally M is just $|W|$ copies of $\mathcal{O}_{\mathfrak{h}}$, and the sections of N then correspond to eM , but the action of differential operators need to be conjugated by the Jacobian of the quotient map, which in this case is δ (c.f. [HS] for a discussion of this in the context of push-forward of \mathcal{D} -modules – this is the reason we must twist by the action of δ in the definition of Ψ_c).

We now consider the other side of the diagram. The construction of the \mathcal{D} -module, \mathcal{M} say, in \mathcal{C}_c from M proceeds via the isomorphism Ψ_c from $e\mathcal{H}_c e$ to $(\mathcal{D}_c(\mathfrak{X})/\mathcal{D}_c(\mathfrak{X})\mathfrak{g}_c)^{\text{SL}(V)}$, so that

$$\mathcal{M} = \mathcal{D}_{\mathfrak{X},c} \otimes_{\Psi_c} eM.$$

On $\mathfrak{G}^{\text{rss}}$ the map Ψ_c discussed in section 3 gives an isomorphism between this algebra and $\mathcal{D}(\mathfrak{h}^{\text{reg}})^W$, where the conjugation by s^c used in the deformed Harish-Chandra homomorphism corresponds the tensor product by $L_{q^{-1}}$ in our construction of $KZ_{(1^n)}$. Then the above discussion shows that we have $j^*(\mathcal{M}) \cong \pi_1^*(N)$, and hence the monodromy measured by $KZ_{(1^n)}$ is exactly that measured by KZ . \square

Remark 5.5.1. A fundamental conjecture due to Kashiwara (a proof of which has recently been announced by Kashiwara and Vilonen [KV]) states that a regular holonomic \mathcal{E}_X -module extends uniquely beyond an analytic subset of codimension greater or equal to three, which means that the structure of the stack of such \mathcal{E}_X -modules is captured in codimension two. Loosely, the “codimension zero” information is given by a collection of local systems, and the codimension 1 information provides certain “glue” between them, with codimension two imposing additional constraints on objects. This implies that one can understand regular holonomic \mathcal{D}_X -modules whose singular support lies in a certain Lagrangian Λ via the geometry of Λ up to codimension two singularities, and in the topological context this idea has been studied in [GMV]. One motivation for this paper came from a desire to get such a “microlocal” understanding of the category \mathcal{C}_c (note that it is shown in [GG, §5.3] that the \mathcal{D} -modules in the category \mathcal{C}_c are regular holonomic – indeed this follows by a standard argument from the finiteness of the number of orbits of $\text{PGL}(V)$ on $\mathcal{N} \times \mathbb{P}(V)$). Note that the topological case and its relation to \mathcal{E}_X -modules has also been studied in [W1], [W2].

6. CHARACTERISTIC CYCLES OF STANDARD MODULES.

6.1. Having defined microlocal KZ functors, we now attempt a first study of them. For this, as for the original KZ functor, we use the standard modules, whose construction we now recall. These modules are analogous to Verma modules in Lie theory: they are defined for any value of c , and are generically simple.

Note that $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$ is a subalgebra of \mathcal{H}_c , and it has a natural evaluation homomorphism $\text{ev}: \mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W] \rightarrow \mathbb{C}[W]$ given by $P \otimes w \mapsto P(0).w$. Thus pull-back via ev allows us to lift modules for $\mathbb{C}[W]$ to modules for $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$. Given

τ a representation of W we define a standard module⁹ $M(\tau)$ by inducing the pull-back of τ to \mathcal{H}_c :

$$M(\tau) = \mathcal{H}_C \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]} \text{ev}^*(\tau).$$

If τ is an irreducible representation indexed by a partition λ , then we write M_λ for $M(\tau)$.

The standard modules have a unique simple quotient L_λ , and by the general machinery of highest weight categories one can also establish a Brauer-Bernstein-Gelfand type reciprocity formula relating the multiplicity of simple modules in standards to the multiplicity of standard modules in projective modules, see [Gu]. It follows easily from this that \mathcal{O}_c is semisimple if and only if the standard modules are simple. We wish to calculate the dimensions of the local systems given by the functors KZ_μ on these modules, or in other words (see §5.3) we wish to calculate the multiplicities of the characteristic cycle of these modules on the corresponding component of Λ .

6.2. We now wish to show that the characteristic cycle of a standard module is independent of the parameter c . To do this we use the “Harish-Chandra” module F_{HC} defined by Gan and Ginzburg [GG, §7.4]. This is the $\mathcal{D}_c(\mathfrak{X})$ -module defined by

$$F_{HC} = \mathcal{D}(\mathfrak{X}, c) / (\mathcal{D}(\mathfrak{X}, c)\mathfrak{g}_c + \mathcal{D}(\mathfrak{X}, c)\mathfrak{Z}_+),$$

where \mathfrak{Z}_+ denote augmentation ideal of \mathcal{Z} the algebra of $\text{ad}(\mathfrak{g})$ -invariant constant coefficient differential operators on \mathfrak{g} . We first show that the characteristic cycle of F_{HC} is independent of c .

Lemma 6.2.1. *The characteristic cycle of F_{HC} is given by the ideal $I = \langle \sigma(x), \sigma(z) : x \in \mathfrak{g}, z \in \mathcal{Z}_+ \rangle$, where σ denotes the symbol map, that is, the generators \mathfrak{g} and \mathcal{Z}_+ are involutive. Thus $\text{CC}(F_{HC})$ is independent of c .*

Proof. To see that our generators are involutive, we use the following criterion [K6, §2.2]: if $\{x_i : 1 \leq i \leq m\}$ are elements of \mathcal{D} of order m_i which satisfy the conditions that

- (1) $\bigcap_{i=1}^m \sigma(x_i)^{-1}(0)$ is of codimension m ;
- (2) $[x_i, x_j] = \sum_k q_{ijk} x_k$, for some $q_{ijk} \in F_{m_i+m_j-m_k-1}$;

then the system of generators is involutive.

In our case, take $\{x_i\}$ to be a basis of \mathfrak{g} (so of order 1) and the standard generators of \mathcal{Z}_+ (of order $2, \dots, n-1$). The second condition is then clearly satisfied, since the operators in \mathfrak{Z}_+ commute with each other and \mathfrak{g}_c . For the first condition, note that the associated graded generators define the set

$$\{(X, Y, v, w) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^* : \mu(X, Y, v, w) = 0, y \text{ nilpotent}\},$$

where μ is the moment map, that is \mathcal{M}_{nil} (in fact with a nonreduced scheme structure), so the first condition is satisfied (thus the lemma reduces essentially to the flatness of the moment map). The independence of c follows immediately. \square

Recall the definition of \mathscr{Y} in §5.4. In the following Lemma we use the fact that we may calculate the characteristic cycle of a \mathcal{D} -module either by taking a good filtration and calculating the multiplicity of the resulting \mathcal{O} -coherent sheaf on the components of its support, or by taking the multiplicities on the components of

⁹This construction of course works for any complex reflection group, not only symmetric groups.

the support of the associated \mathcal{E} -module. See e.g. [K6, Chapter 7] for a discussion of this fact at the level of supports.

Lemma 6.2.2. *Suppose that $c \notin \mathcal{V}$. Let M_λ be a standard module, and μ a partition of n . Then $\dim(KZ_\mu(M_\lambda))$ is independent of c .*

Proof. Let $\text{co}(\mathfrak{h}^*)$ be the coinvariant module $\mathbb{C}[\mathfrak{h}^*]/\langle I \rangle$, where I is the augmentation ideal in $\text{Sym}(\mathfrak{h})^W$, viewed as a representation of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ in the obvious way, and let C denote the \mathcal{H}_c module induced from $\text{co}(\mathfrak{h})$. It is shown in [GG, §7] that if $\langle I \rangle$ denotes the ideal of $e\mathcal{H}_ce$ generated by $\text{Sym}(\mathfrak{h})_+^W$, then $eC \cong e\mathcal{H}_ce/\langle I \rangle$. Using this, they show that $\mathbb{H}(F_{HC}) = eC$. Now the discussion in §5.4 shows that if $c \notin \mathcal{V}$ then the intersection of the characteristic cycle of F_{HC} with U is completely determined by $\mathbb{H}(F_{HC})$, and thus it coincides with that of $\text{Loc}(eC)$, (as $\mathbb{H} \circ \text{Loc}$ is naturally isomorphic to the identity). It follows that the characteristic cycle of $\text{Loc}(eC)$ intersected with U is independent of c .

Now consider $R = \Delta(\mathbb{C}[W])$, the representation of \mathcal{H}_c induced from the regular representation of W . Since it is clear that we may filter $\text{co}(\mathfrak{h}^*)$ in such a way that the associated graded module is isomorphic to $\mathbb{C}[W]$ as a $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ -module, and $e\mathcal{H}_c$ is a flat right $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ -module, it follows that R and C have the same class in the Grothendieck semigroup for \mathcal{O}_c , and hence the same characteristic cycle.

Since R is clearly finitely generated as a $\mathbb{C}[\mathfrak{h}]$ -module, and $\mathbb{C}[\mathfrak{h}]$ is finite over $\mathbb{C}[\mathfrak{h}]^W$, the module $\text{Loc}(eR) = \mathcal{L}_c \otimes_{e\mathcal{H}_ce} eR$ is finite over $\mathbb{C}[\mathfrak{X}]^G$. It follows that if Γ_i is a good filtration on \mathcal{L}_c , the filtration $\Gamma_i \otimes eR$ is good for $\text{Loc}(eR)$. Thus we may use the order filtration on $\mathcal{D}_c(\mathfrak{X})$ to obtain a good filtration on \mathcal{L}_c .

Now $\mathbb{C}[W]$ is of course a bimodule for W , hence it carries a right action of W which commutes with the left action used in the construction of R . Thus we have a right W action on eR via $e\mathcal{H}_ce$ -module automorphisms, and so the localization $\text{Loc}(eR)$ carries a W -action as a $\mathcal{D}_{\mathfrak{X},c}$ -module, which is compatible with our filtration. But now taking the isotypic components according to this action yields the standard modules, and hence their characteristic cycles are determined by this action on $\text{CC}(\text{Loc}(eR)) = \text{CC}(\text{Loc}(eC))$, which must therefore be independent of c as required. \square

6.3. The previous lemma shows that, in order to calculate the part of the characteristic cycles of $\text{Loc}(eM_\lambda)$ lying in Λ^s we may assume that $c = 0$. It is known [BEG] that the standard modules are simple whenever the finite Hecke algebra at $q = e^{2\pi ic}$ is semisimple. Thus it follows the standard modules are simple when $c = 0$. We will write \mathfrak{M}_λ for the \mathcal{D} -module corresponding to the standard module eM_λ when $c = 0$. We now wish to identify \mathfrak{M}_λ more explicitly.

Recall the Grothendieck simultaneous resolution $\mu: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ where

$$\tilde{\mathfrak{g}} = \{(x, F) \in \mathfrak{g} \times \mathcal{B} : F = (F_i)_{1 \leq i \leq n}, x(F_i) \subseteq F_i\}$$

and μ is the projection to the first factor. Over the subset \mathfrak{g}^{rss} of regular semisimple elements of \mathfrak{g} this is a W -covering. Moreover, it is known by [HK, §4] that the push-forward $F_{\tilde{\mathfrak{g}}} = \mu_*(\mathcal{O}_{\tilde{\mathfrak{g}}})$ is a semisimple holonomic \mathcal{D} -module which is the minimal extension of its restriction to \mathfrak{g}^{rss} , where it is just the local system corresponding to the principal W -covering. For λ an irreducible representation of W , let $\mathfrak{M}_\lambda^{\mathfrak{g}}$ denote the simple $\mathcal{D}_{\mathfrak{g}}$ -module corresponding to the minimal extension of the local system given by λ on \mathfrak{g}^{rss} . Since we are assuming that $c = 0$, the sheaf of rings $\mathcal{D}_{\mathfrak{X},0}$ is just

$\mathcal{D}_{\mathfrak{X}} = \mathcal{D}_{\mathfrak{g}} \boxtimes \mathcal{D}_{\mathbb{P}(V)}$, and we may pull-back $F_{\mathfrak{g}}$ to a $\mathcal{D}_{\mathfrak{X}}$ -module $F = F_{\mathfrak{g}} \boxtimes \mathcal{O}_{\mathbb{P}(V)}$. It follows that F is a direct sum of simple modules $\mathfrak{M}_{\lambda}^{\mathfrak{g}} \boxtimes \mathcal{O}_{\mathbb{P}(V)}$.

Lemma 6.3.1. *The sheaf \mathfrak{M}_{λ} arising from the standard module M_{λ} at $c = 0$ is isomorphic to $\mathfrak{M}_{\lambda}^{\mathfrak{g}} \boxtimes \mathcal{O}_{\lambda}$.*

Proof. We use the (original) KZ -functor. It is known by [GGOR, §6.2] that the KZ -functor applied to a standard module M_{λ} yields the corresponding cell module S_{λ} for the finite Hecke algebra. Since we are assuming $c = 0$ this in turn is just the corresponding irreducible representation of W . Thus using Proposition 5.5.1 it follows that $\mathfrak{M}_{\lambda|\mathfrak{X}^{rss}}$ is just the local system given by λ . Since \mathfrak{M}_{λ} is simple, it follows that the sheaf \mathfrak{M}_{λ} is just the minimal (or Goresky-MacPherson) extension of this local system to \mathfrak{X} . However, the sheaf $\mathfrak{M}_{\lambda}^{\mathfrak{g}} \boxtimes \mathcal{O}_{\mathbb{P}(V)}$ clearly restricts to give the same local system and is also simple, hence we see that $\mathfrak{M}_{\lambda} = \mathfrak{M}_{\lambda}^{\mathfrak{g}}$ as required. \square

6.4. To compute the characteristic cycles of the standard modules, we will actually use a Fourier dual description of the variety Λ . Thus we need to use the following standard remark: let $X = E \times B$ be a trivial vector bundle over B and let $\mathcal{D}_X^{\text{mon}}$ be the category of monodromic \mathcal{D} -modules on X , that is, those for which the action of the Euler vector field associated to E is locally finite. Then it is known that the partial Fourier transform \mathcal{F}_E along E gives an equivalence between $\mathcal{D}_{E \times B}^{\text{mon}}$ and $\mathcal{D}_{E^* \times B}^{\text{mon}}$.

Lemma 6.4.1. *We may canonically identify*

$$T^*(E \times B) \cong T^*(E^* \times B) \cong E \times E^* \times T^*B.$$

Under this isomorphism, if N is a monodromic \mathcal{D}_X -module we have

$$\text{CC}(N) \cong \text{CC}(\mathcal{F}_E(N))$$

Proof. For a discussion of this see for example [HK, §3]. \square

We apply this to $\mathfrak{X} = \mathfrak{g} \times \mathbb{P}(V)$ with $E = \mathfrak{g}$ and $B = \mathbb{P}(V)$. The isomorphism $T^*(\mathfrak{g} \times \mathbb{P}(V)) \cong T^*(\mathfrak{g}^* \times \mathbb{P}(V))$ is simply given by identifying both with $\mathfrak{g} \times \mathfrak{g}^* \times T^*\mathbb{P}(V)$ (or via the trace form $\mathfrak{g} \times \mathfrak{g} \times T^*\mathbb{P}(V)$). We want to understand the components of Λ in this “Fourier dual” picture. This is essentially already done in [GG] but note they use a different transform – the bundle maps we use are given by

$$\begin{aligned} p: T^*(\mathfrak{g} \times \mathbb{P}(V)) &\rightarrow \mathfrak{g} \times \mathbb{P}(V), & (X, Y, i, j) &\mapsto (X, i) \\ \check{p}: T^*(\mathfrak{g}^* \times \mathbb{P}(V)) &\rightarrow \mathfrak{g}^* \times \mathbb{P}(V), & (X, Y, i, j) &\mapsto (Y, i). \end{aligned}$$

whereas in [GG] they use $\check{p}(X, Y, i, j) = (Y, j)$. Nevertheless the same argument they use gives the following:

Lemma 6.4.2. *Let Λ^* denote the image of Λ under the canonical isomorphism in $T^*(\mathfrak{g}^* \times \mathbb{P}(V))$. The Λ^* is the union of the conormal bundle to the $\text{PGL}(V)$ -orbits on $\mathcal{N} \times \mathbb{P}(V)$.*

Proof. This follows from the fact that any conic Lagrangian in a cotangent bundle T^*X is the closure of the conormal bundle of a smooth locally closed subvariety of X , and the fact that $\text{PGL}(V)$ acts with finitely many orbits on $\mathcal{N} \times \mathbb{P}(V)$. \square

Thus the \mathcal{D} -modules on $\mathfrak{g}^* \times \mathbb{P}(V)$ which we study are supported on $\mathcal{N} \times \mathbb{P}(V)$ and are smooth along $\text{PGL}(V)$ -orbits. Note that since there are finitely many orbits, they give a Whitney stratification of $\mathcal{N} \times \mathbb{P}(V)$. For our purposes we need to check which $\text{PGL}(V)$ -orbits correspond to the components in Λ^s .

Lemma 6.4.3. *The component Λ_λ^s ($\lambda \vdash n$) of Λ^s corresponds to the $\mathrm{PGL}(V)$ -orbits*

$$\mathcal{O}_\lambda^{\mathrm{cyc}} = \{(Y, \ell) : Y \text{ has Jordan type } \lambda, \text{ and } \mathfrak{g}_Y \cdot \ell = V\},$$

where \mathfrak{g}_Y denotes the centralizer of Y in \mathfrak{g} .

Proof. Recall from the last paragraph of Section 4.2 that Λ_λ^s corresponds to the component $\Lambda_{\lambda, \emptyset}$ of Λ , that is the closure of the conormal bundle $T_S^*(\mathfrak{X})$ where S is the stratum of pairs (X, ℓ) where X has eigenvalues with multiplicity given by λ and ℓ is cyclic for X . Then the proof of Lemma 4.2.2 shows that the set of nilpotent endomorphisms Y which commute with X , have Jordan type λ , and for which the projection of ℓ to each generalized eigenspace of X is cyclic for the restriction of Y to that space, is open dense in $\check{p}(\Lambda_{\lambda, \emptyset})$. Since \mathfrak{g}_Y contains $\mathbb{C}[X, Y]$ it follows $\mathfrak{g}_Y \cdot \ell = V$ for such Y , and these clearly form a single $\mathrm{PGL}(V)$ orbit. \square

Note that if \mathcal{O}_λ denotes the orbit of nilpotent endomorphisms of Jordan type λ , then clearly

$$\mathcal{O}_\lambda^{\mathrm{cyc}} = \Lambda^s \cap \mathcal{O}_\lambda \times V.$$

where we have identified Λ^s with a subset of Λ^* using the isomorphism above.

6.5. Since the characteristic cycle of \mathfrak{M}_λ is determined by its deRham complex $\mathrm{DR}(\mathfrak{M}_\lambda)$ we may work purely topologically.

Theorem 6.5.1. *Suppose that $c \notin \mathscr{Y}$. Let $\mathfrak{M}_{\lambda, c}$ be the $\mathcal{D}_{\mathfrak{X}, c}$ -module associated to the standard module M_λ . Then we have*

$$\mathrm{CC}(\mathfrak{M}_{\lambda|_{\mathfrak{X}^{\mathrm{reg}}}}) = \sum_{\mu \leq \lambda} K_{\lambda, \mu} [\Lambda_\mu],$$

where \leq denotes the dominance order, and $K_{\lambda, \mu}$ are the Kostka numbers. In other words, we have

$$\dim(KZ_\mu(eM_\lambda)) = K_{\lambda, \mu}.$$

Proof. We have already seen that it is enough to compute the characteristic cycle in the case $c = 0$, where we write \mathfrak{M}_λ instead of $\mathfrak{M}_{\lambda, 0}$. Taking the Fourier transform $\mathcal{F}_g(\mathfrak{M}_\lambda)$ we obtain a \mathcal{D} -module on $\mathcal{N} \times \mathbb{P}(V)$. From the preceding discussion we know that $\mathfrak{M}_\lambda = \mathfrak{M}_\lambda^g \boxtimes \mathcal{O}_{\mathbb{P}(V)}$, and its Fourier transform is then just $\mathcal{F}_g(\mathfrak{M}_\lambda^g) \boxtimes \mathcal{O}_{\mathbb{P}(V)}$. Now by [HK, §5] the \mathcal{D} -module $\mathcal{F}_g(\mathfrak{M}_\lambda^g)$ corresponds to the intersection cohomology sheaf \mathcal{I}_λ on \mathcal{O}_λ , so that we are reduced to calculating the characteristic cycles of intersection cohomology sheaves of nilpotent orbits. But these are known thanks to [EM] which uses a torus symmetry trick and the knowledge of the stalk Euler characteristics of the complexes. Indeed [EM, Lemma 3.2] shows that for nilpotent orbits $\alpha, \beta \subset \mathcal{N}$ (in type A) the local Euler obstruction $c_{\alpha, \beta}$ is zero whenever $\alpha \neq \beta$ (and by definition it is 1 when $\alpha = \beta$). Thus since our intersection cohomology sheaves \mathcal{I}_λ are constructible with respect to the stratification of \mathcal{N} it follows from the Index Theorem (see Theorem 9.2.1) that

$$\mathrm{CC}(\mathcal{I}_\lambda) = \sum_{\mu} \chi_\mu(\mathcal{I}_\lambda) [T_{\mathcal{O}_\mu}^* \mathfrak{g}],$$

where $\chi_\mu(\mathcal{I}_\lambda)$ denotes the stalk Euler characteristic of \mathcal{I}_λ along the orbit \mathcal{O}_μ . But it is shown in [L, §2] that the dimensions of the stalk cohomologies of the intersection cohomology sheaves of nilpotent orbits in $\mathrm{GL}(V)$ are given by Kostka polynomials, from which it follows immediately from this that the stalk Euler characteristics are given by the Kostka numbers $K_{\lambda, \mu}$, which completes the proof. \square

Remark 6.5.1. Note that the real advantage of the description of $\text{Loc}(eM_\lambda)|_{c=0}$ as $\mathfrak{M}_\lambda^g \boxtimes \mathcal{O}_{\mathbb{P}(V)}$ was that it makes the computation of the Fourier transform straightforward. For details on the index theorem which reduce the computation of characteristic cycles to that of Euler characteristics and the local Euler obstructions see the Appendix.

Remark 6.5.2. In [GS, §2.7] Gordon and Stafford defined a characteristic cycle for modules in category $\mathcal{O}_c^{\text{sp}}h$ via their \mathbb{Z} -algebra construction, and computed them for standard modules. Subsequently, Ginzburg, Gordon and Stafford [GGS] established the equivalence of their definition with the \mathcal{D} -module one given in [GG], which implies the above result. Their calculation however depended on Haiman's work on the Hilbert scheme, while the above calculation uses only standard tools from \mathcal{D} -modules and the elegant torus action trick of Evens-Mirkovic.

7. FACTORIZATION THROUGH HECKE ALGEBRAS

7.1. We now wish to use \mathcal{D} -module techniques to show that the KZ -functor factors through a functor to representations of the finite Hecke algebra. Thus we wish to check that the generators of the braid group satisfy the quadratic relation

$$(T_s - 1)(T_s + q) = 0.$$

Since this relation involves only a single generator, we first establish it in the rank one case. For this we need to recall the classification of regular singularities \mathcal{D} -modules on the complex line which are \mathcal{O} -coherent on $\mathbb{C} - \{0\}$. Let \mathcal{M} be such a \mathcal{D} -module. Then by taking a V -filtration of $V^\bullet \mathcal{M}$ (see for example [K3]) one may define functors of nearby and vanishing cycles:

$$\Psi(\mathcal{M}) = \text{gr}_0^V(\mathcal{M}), \quad \Phi(\mathcal{M}) = \text{gr}_{-1}^V(\mathcal{M}),$$

from \mathcal{D} -modules to finite-dimensional vector spaces. Moreover these functors are equipped with a “canonical” map $c: \Psi(\mathcal{M}) \rightarrow \Phi(\mathcal{M})$ (induced by the action of ∂), and a “variation” map $v: \Phi(\mathcal{M}) \rightarrow \Psi(\mathcal{M})$. These are related to each other by the equations:

$$(7.1) \quad c \circ v = T - \text{Id}, \quad v \circ c = T - \text{Id}.$$

where T is the monodromy operator $\exp(-2i\pi x\partial)$. In fact it can be shown that our category of \mathcal{D} -modules is then equivalent to the category of pairs of vector spaces (V, W) together with linear maps $c: V \rightarrow W$, and $v: W \rightarrow V$ such that 1 is not an eigenvalue of either cv or vc .

Lemma 7.1.1. *Suppose that $n = 2$. The monodromy representations of the braid group $\mathcal{B}_2 = \mathbb{Z}$ given by $KZ_{(1^2)}$ factor through the Hecke algebra at $q = e^{2\pi ic}$.*

Proof. By taking co-ordinate “centre of mass” coordinates x, x', y, y' such that $W = S_2$ acts trivially on x', y' and by the sign representation on x, y , we may reduce to the case of $T^*\mathbb{P}^1$. By the discussion in the last section of [KR], the sheaf of rings on $\text{Hilb}^2(T^*\Sigma)$ localizing $e\mathcal{H}_ce$ when restricted to the exceptional divisor is just the twisted ring of differential operators $\mathcal{D}_{\mathbb{P}^1, \lambda}$ where $\lambda = c - \frac{1}{2}$. The characteristic cycle of a module \mathcal{M} in \mathcal{C}_c must then lie in $\mathbb{P}^1 \cup T_0^*\mathbb{P}^1$, and \mathcal{M} becomes a module for the sheaf of rings $\mathcal{D}_{\mathbb{P}^1, \lambda}$. We claim that this implies the monodromy around $T_0^*\mathbb{P}^1 - \{0\}$, which is the monodromy given by $KZ_{(1^2)}$, satisfies the quadratic relation of the Hecke algebra.

In terms of the sheaf of rings $\mathcal{D}_{\mathbb{P}^1, \lambda}$, this monodromy is the action of T on the vanishing cycles sheaf. Since the monodromy operator is given by $\exp(2\pi i x \partial_x)$, the twisted equivariance forces T to act as $e^{2\pi i \lambda} = -e^{2\pi i c} = -q$ on the nearby cycles sheaf. But now it follows that $v \circ c = -q - 1$. By Equation 7.1 we see that this implies that on the vanishing cycles we have

$$(T - \text{Id})^2 = (c \circ v)^2 = c \circ (v \circ c) \circ v = c \circ (-q - 1) \circ v = -(q + 1)(T - \text{Id}).$$

Rearranging this equation gives us the Hecke algebra quadratic equation as required. \square

7.2. We now deduce the general case by reduction to the rank one case. Let \mathfrak{g}_i denote the subset of \mathfrak{g} consisting of matrices of the form

$$\begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ \vdots & 0 & A & 0 & \vdots \\ \vdots & \vdots & 0 & a_{i+2} & 0 \\ 0 & \cdots & \cdots & 0 & \ddots \end{pmatrix}$$

where the a_j ($j \in \{1, \dots, i-1, i+2, \dots, n\}$) are pairwise distinct numbers, A is a 2×2 matrix with entries in the i th and $(i+1)$ th rows, and the eigenvalues of A are distinct from the a_j s. (The set \mathfrak{g}_1 is also used to reduce to rank 1 in [KR, §3].) It is easy to check that $\mathfrak{g}_i \times V$ is noncharacteristic for Λ , thus if M is an object in \mathcal{C}_c , its pull-back to \mathfrak{g}_i is holonomic with characteristic variety contained in the fibrewise projection of Λ to $T^*(\mathfrak{g}_i \times V)$. More precisely, letting $i: \mathfrak{g}_i \times V^\circ \rightarrow \mathfrak{G}^\circ$ denote the inclusion map we have the diagram:

$$\begin{array}{ccc} T^*(\mathfrak{g}_i \times V^\circ) & \xleftarrow{i_d} & \mathfrak{X} \times_{\mathfrak{g}_i \times V^\circ} T^*(\mathfrak{g}_i \times V^\circ) \\ & & \downarrow i_\pi \\ & & T^*\mathfrak{G}^\circ \end{array}$$

and $\text{Ch}(i^*(M)) \subset i_d i_\pi^{-1}(\text{Ch}(M))$. Here i_π is just the obvious inclusion map, and i_d is the fibrewise projection map. Now it is immediate from the definitions that $i_d i_\pi^{-1}(\Lambda)$ can be identified with $\Lambda_2 \times \mathbb{C}^{n-2} \times (\mathbb{C}^\times)^{n-2}$, where Λ_2 is the variety Λ in the case $n = 2$. Moreover this identification is equivariant for the action of the group $G_2 = T \times \text{SL}_2$ where G_2 is embedded in $\text{GL}(V)$ in the obvious way so as to be compatible with the map i . Let

$$A(t) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix}$$

Picking pairwise distinct complex numbers (a_j) none of which lie in $\{\pm 1\}$, the action of the generator T_i on $KZ_{(1^n)}(M)$ is given by monodromy around the curve $t \mapsto i(a_1, \dots, a_{i-1}, A(t), a_{i+2}, \dots, a_n)$. Thus we see that it is enough to check the quadratic relation holds for the case $n = 2$, which has already been checked. Thus we have the following theorem

Theorem 7.2.1. *Let $M \in \mathcal{C}_c$. The monodromy representation given by $KZ_{(1^n)}(M)$ factors through the Hecke algebra $H_W(q)$ where $q = e^{2\pi i c}$.*

This recovers, in type A , Theorem 5.13 of [GGOR] which is one of the central results of that paper. The proof given there is quite different – it checks the result directly on standard modules, and then deduces the general case by a deformation argument.

Remark 7.2.1. The result of this section applies only to the original KZ -functor. At the moment the author does not have an analogue of this result for the microlocal KZ -functors. One reason for this is that it is harder to give a presentation of the groups \mathcal{B}_λ . However since \mathcal{C}_c is Artinian it follows formally that the action of \mathcal{B}_λ will factor through a finite-dimensional algebra. Note that Rouquier [R] has shown that the category \mathcal{O}_c is equivalent to the category of representations of the q -Schur algebra. Since the q -Schur algebra is a quotient of the quantum group $U_v(\mathfrak{gl}_n)$, its representations carry Lusztig's braid group action, and the partitions λ can be interpreted as weights of such representations. The subgroup of Lusztig's braid group acting on the weight space corresponds to \mathcal{B}_λ and we conjecture that given M a representation in \mathcal{O}_c , the representation KZ_λ is given by the λ -weight space of the corresponding representation of the q -Schur algebra equipped with this braid group action.

8. ON THE MINIMAL RESOLUTION OF \mathbb{C}^2/μ_l AS A HYPERTORIC VARIETY

8.1. The ideas of this paper should have applications to a larger class of rational Cherednik algebras than that which is attached to the symmetric groups. To illustrate this, in this section we consider the case of the rational Cherednik algebra \mathcal{H}_κ attached to a cyclic group μ_l of order l , which we regard as a complex reflection group (with the obvious one-dimensional representation). We use quantum Hamiltonian reduction on a moduli space of representations for the cyclic quiver as in [Ku] and [BK], which give a deformation of the minimal resolution of a Kleinian singularity. Each components of the exceptional divisor of the resolution yields an exact functors on category \mathcal{O} for \mathcal{H}_κ , and moreover, taken together, they can be used to construct a functor from \mathcal{O}_c to the category of representations of the “cyclotomic q -Schur algebra” (i.e. the quasi-hereditary cover of [DJM, §6]) of the corresponding cyclotomic Hecke algebra. This gives a geometric perspective on the the ideas of [GGOR] and [R]. Indeed it is shown in [GGOR, Theorem 5.16] that category \mathcal{O}_c can be viewed as a quasi-hereditary cover of the category of representations of the Hecke algebra (attached to any complex reflection group) while our construction yields an explicit functor from category \mathcal{O}_c to the cyclotomic q -Schur algebra which is known to be a quasi-hereditary cover of the Hecke algebra.

We begin by recalling a construction of the algebra \mathcal{H}_κ . Let μ_l denote the cyclic group of order l (thought of as the l -th roots of unity in \mathbb{C} , i.e. as a complex reflection group with a single “hyperplane” $\{0\}$). The action of μ_l on \mathbb{C} induces an action on $\mathcal{D}(\mathbb{C}^\times)$, so we may consider the smash product $\mathcal{D}(\mathbb{C}^\times) \rtimes \mathbb{C}[\mu_l]$ of $\mathcal{D}(\mathbb{C}^\times)$ with the group algebra of μ_l . To clarify the notation we fix a generator γ of μ_l and write elements of $\mathbb{C}[\mu_l]$ in the form $\sum_{i=0}^{l-1} a_i \gamma^i$ where $a_i \in \mathbb{C}$ ($0 \leq i \leq l-1$). Then if $\kappa = (\kappa_0, \dots, \kappa_{l-1}) \in \mathbb{C}^l$ we define a Dunkl operator by the formula

$$\partial_\kappa = \frac{d}{dz} + \frac{l}{z} \sum_{i=0}^{l-1} \kappa_i e_i,$$

where the e_i denote the idempotents of $\mathbb{C}[\mu_l]$, that is $e_i = (1/l) \sum_{j=0}^{l-1} \zeta^{ij} \gamma^j$, for $i = 0, 2, \dots, l-1$. The algebra \mathcal{H}_κ is then the subalgebra of $\mathcal{D}(\mathbb{C}^\times) \rtimes \mathbb{C}[\mu_l]$ generated by z, ∂_κ and γ .

8.2. We now review the description of the minimal resolution X of \mathbb{C}^2/μ_l as a hypertoric variety. Let Q be the cyclic quiver with l vertices $I = \{0, 1, \dots, l-1\}$ (read as elements of $\mathbb{Z}/l\mathbb{Z}$), and edges $H = \{h_i : 1 \leq i \leq l\}$, where h_i is the edge $(i-1) \rightarrow i$. Let \bar{Q} denote the quiver (I, E) where $E = H \cup \bar{H}$ where $\bar{H} = \{\bar{h}_i : 1 \leq i \leq l\}$ and \bar{h}_i denotes the edge from $i \rightarrow (i-1)$, i.e. the edge in the opposite direction to h_i . We will denote the initial and terminal vertex of an edge $h \in E$ by $s(h)$ and $t(h)$ respectively.

A representation of \bar{Q} is an I -graded vector space $V = \bigoplus_{i=0}^{l-1} V_i$ together with an element of $E_{V, \bar{H}} = \bigoplus_{h \in \bar{H}} \text{Hom}(V_{s(h)}, V_{t(h)})$, so that the moduli space of representations of \bar{Q} (of a fixed graded dimension) may be identified with the quotient of $E_{V, \bar{H}}$ by the action of $G_V = \prod_{i \in I} \text{GL}(V_i)/\mathbb{G}_m$ where \mathbb{G}_m is the diagonal copy of scalar operators (which acts trivially on $E_{V, \bar{H}}$). Similarly, we have the moduli space of representations of Q given by the quotient of G_V acting on $E_{V, H}$.

8.3. We will be interested in the case where V has graded dimension $\delta = (1, 1, \dots, 1)$, and thus we assume $\dim(V_i) = 1$ in what follows. We will also write $G = G_V \cong (\mathbb{G}_m)^l/D$ (where D is the diagonal \mathbb{G}_m), a rank $l-1$ torus, with Lie algebra $\mathfrak{g} = \mathbb{C}^l/\mathbb{C}$. Note that character lattice $X(G)$ of G may be viewed as the set of $\theta = (\theta_0, \theta_2, \dots, \theta_{l-1}) \in \mathbb{Z}^l$ such that $\sum_{i=0}^{l-1} \theta_i = 0$.

Definition 8.3.1. Given $\theta \in X(G)$, we say a representation (V, x) of \bar{Q} is θ -semistable if for any subrepresentation W of V we have $\sum_{i=0}^{l-1} \theta_i \dim(W_i) \leq 0$. We set \tilde{X}_θ to be the subset of $E_{V, \bar{H}}$ consisting of θ -semistable representations, a G -invariant subset of E_V .

Now $E_V \cong T^*E_{V, H}$ since we may identify $\text{Hom}(V_{i-1}, V_i)$ with the dual of $\text{Hom}(V_i, V_{i-1})$ via the trace pairing $(x, y) \mapsto \text{tr}_{V_i}(yx)$. Therefore $E_{V, \bar{H}}$ is a symplectic vector space, and the action of G clearly preserves the symplectic form. We will write $(a_i, b_i)_{1 \leq i \leq l}$ for a point in $E_{V, \bar{H}}$ where $a_i \in \text{Hom}(V_{i-1}, V_i)$ and $b_i \in \text{Hom}(V_i, V_{i-1})$. Since \tilde{X}_θ is clearly a G -invariant open subset of $E_{V, \bar{H}}$, it is a complex symplectic manifold acted on by G . The associated moment map $\mu: \tilde{X}_\theta \rightarrow \mathfrak{g}$ is given by:

$$(a_i, b_i)_{1 \leq i \leq l} \mapsto (a_{i+1}b_{i+1} - a_i b_i)_{1 \leq i \leq l}.$$

Let μ_l act on \mathbb{C}^2 via the action $\gamma(x, y) = (\gamma x, \gamma^{-1}y)$ for $\gamma \in \mu_l$, and write \mathbb{C}^2/μ_l for the quotient, a rational surface singularity of type A . Given $(a_i, b_i)_{1 \leq i \leq l} \in E_{V, \bar{H}}$, a pair $(\bar{a}, \bar{b}) \in \mathbb{C}^2$ satisfying the conditions $\bar{a}^l = a_1 a_2 \dots a_l$ and $\bar{b}^l = b_1 b_2 \dots b_l$ and $\bar{a}\bar{b} = a_1 b_1$ is well defined up to the action of μ_l . This yields a map from $E_{V, \bar{H}}$ to \mathbb{C}^2/μ_l which induces an isomorphism of varieties between $E_{V, \bar{H}}/G$ and \mathbb{C}^2/μ_l .

The G.I.T. quotient of \tilde{X}_θ by the action of G is denoted X_θ (thus in particular $X_0 = E_{V, \bar{H}}/G$). We write $[a_i, b_i]_{1 \leq i \leq l}$ for the equivalence class in X_θ of a point $(a_i, b_i)_{1 \leq i \leq l}$ in \tilde{X}_θ . When (θ_i) satisfies $\sum_{k=i}^{j-1} \theta_k \neq 0$ for all $i, j \in \mathbb{Z}/l\mathbb{Z}$ where $i \neq j$, this quotient is a smooth surface, and the map π to the affine quotient $E_{V, \bar{H}}/G$ is a resolution of singularities, in fact the minimal resolution. From now on we will assume this condition on θ holds.

8.4. The varieties X_0 and X_θ are both toric, where the action of $T = \mathbb{G}_m^2$ is induced by the action of T on $E_{V,\bar{H}}$ given by $z \cdot (a_i, b_i)_{1 \leq i \leq l} = (z_1 a_i, z_2 b_i)_{1 \leq i \leq l}$ where $z = (z_1, z_2) \in T$. To describe the structure of X_θ as a toric variety, we need to introduce an ordering on the elements of $\mathbb{Z}/l\mathbb{Z}$ induced by the stability θ .

Definition 8.4.1. We say that $i \triangleright j$ if $\theta_i + \theta_{i+1} + \dots + \theta_{j-1} < 0$. By our condition on θ this is a total order, and we write η_i for the permutation of $\{1, 2, \dots, l\}$ which this yields, i.e. $\eta_1 \triangleright \eta_2 \triangleright \dots \triangleright \eta_l$.

As a toric variety X_θ can be written naturally as a union $X_\theta = \bigcup_{i=1}^l X_i$ where

$$X_i = \{[a_j, b_j]_{1 \leq j \leq l} : a_{\eta_j} \neq 0 \text{ for } j < i; b_{\eta_j} \neq 0 \text{ for } j > i\}$$

and each X_i is an affine toric variety containing a unique T -fixed point p_i with $a_{\eta_i} = b_{\eta_i} = 0$. In fact $X_i \cong T^*\mathbb{C}$, which we may see explicitly as follows: if we pick a basis for the lines $(V_i)_{0 \leq i \leq l-1}$ then $E_{V,\bar{H}}$ may be identified with \mathbb{C}^{2l} where we will still write a point of \tilde{X}_θ as $(a_i, b_i)_{1 \leq i \leq l}$. Let $\bar{x}_i((a_j, b_j)_{1 \leq j \leq l}) = a_i$ and $\bar{y}_i((a_j, b_j)_{1 \leq j \leq l}) = b_i$ be the obvious coordinate functions. Then $X_i = \text{Spec}(R_i)$ where R_i is the subalgebra of function field $\mathbb{C}(\bar{x}_i, \bar{y}_i)_{1 \leq i \leq l}$ generated by the algebraically independent elements

$$\bar{f}_i = \frac{\bar{x}_{\eta_1} \bar{x}_{\eta_2} \dots \bar{x}_{\eta_i}}{\bar{y}_{\eta_{i+1}} \bar{y}_{\eta_{i+2}} \dots \bar{y}_{\eta_l}}, \quad \bar{g}_i = \frac{\bar{y}_{\eta_i} \bar{y}_{\eta_{i+1}} \dots \bar{y}_{\eta_l}}{\bar{x}_{\eta_1} \bar{x}_{\eta_2} \dots \bar{x}_{\eta_{i-1}}}.$$

Note that $\bar{f}_i \bar{g}_i = \bar{x}_{\eta_i} \bar{y}_{\eta_i}$. The subvarieties X_i and X_{i+1} intersect in a complex line whose closure in X_θ we denote by D_i . For $1 \leq i < l$ this closure is a \mathbb{P}^1 and their union is the exceptional divisor of the resolution π . If we let \mathcal{L} denote the preimage of $\pi^{-1}(\{(\bar{a}, \bar{b}) \in X_0 : \bar{a}\bar{b} = 0\})$ then \mathcal{L} is the union of $l+1$ irreducible components D_i , ($0 \leq i \leq l$), where $D_0 \cong D_l \cong \mathbb{C}$, and the remaining D_i are the above \mathbb{P}^1 s. These D_i are all the T -stable divisors in X_θ .

8.5. We now recall the construction of \mathcal{W} -algebras with an F -action on X_θ following [KR]. Let $c = (c_i)_{0 \leq i \leq l-1}$ be an l -tuple of complex numbers such that $\sum_{i=0}^{l-1} c_i = 0$. The \mathcal{W} -algebra \mathcal{A}_c is obtained by quantum Hamiltonian reduction from the standard \mathcal{W} -algebra on $\mathbb{C}^{2l} = T^*\mathbb{C}^n$, whose generators we denote by x_i, y_i ($1 \leq i \leq l$). Clearly \tilde{X}_θ carries the sheaf of algebras $\mathcal{W}_{\tilde{X}_\theta}$, the restriction of the sheaf $\mathcal{W}_{T^*\mathbb{C}^n}$, and we may define the $\mathcal{W}_{\tilde{X}_\theta}$ -module \mathcal{L}_c by

$$\mathcal{L}_c = \mathcal{W}_{\tilde{X}_\theta} / \left(\sum_{i=1}^l \mathcal{W}_{\tilde{X}_\theta} (x_{i+1} y_{i+1} - x_i y_i + \hbar c_i) \right).$$

If $p: \mu^{-1}(0) \rightarrow X_\theta$ denotes the quotient map, then $\mathcal{A}_c = (p_*(\text{End}_{\mathcal{W}_{\tilde{X}_\theta}}(\mathcal{L}_c))^G)^{\text{op}}$ is a \mathcal{W} -algebra on X_θ . The sheaf of algebras $\tilde{\mathcal{A}}_c$ obtained by adjoining a square root of \hbar has an F -action in the sense of [KR, 2.3] of weight one for the \mathbb{C}^\times action giving x_i and y_i weight one and \hbar weight two. We then consider the category $\text{Mod}_F^{\text{good}}(\tilde{\mathcal{A}}_c)$ of good $\tilde{\mathcal{A}}_c$ -modules with an F -action as in [KR, 2.4]. It is known by the work of Holland [H] and Bellamy-Kuwabara [BK, §6] that the algebra $A_c = \text{End}_{\text{Mod}_F^{\text{good}}(\tilde{\mathcal{A}}_c)}(\tilde{\mathcal{A}}_c)^{\text{opp}}$ is isomorphic to the spherical rational Cherednik algebra $U_\kappa = e_0 H_\kappa e_0$ (where e_0 is the idempotent corresponding to the trivial representation of μ_ℓ) and moreover [BK] proves a localization theorem if $i \triangleright j$ whenever $c_i + c_{i+1} + \dots + c_{j-1} \in \mathbb{Z}_{\geq 0}$ (though we do not need this for our constructions, which

only use the existence of the localization functor). Here the deformation parameter $c = c(\kappa)$ is related to the parameter κ via

$$c_i = \kappa_{i+1} - \kappa_i + 1/l - \delta_{i,0}.$$

8.6. Let \mathcal{O}_κ denote the category of representations of the spherical rational Cherednik algebra on which ∂_κ acts locally nilpotently, and let $\mathcal{O}_\kappa^{\text{sph}}$ denote the corresponding category of representations for U_κ . Via localization, this category corresponds to a category of modules for $\tilde{\mathcal{A}}_c$, which we will denote by \mathcal{O}_c . It follows from results of [Ku, §5] that the modules in \mathcal{O}_c are all supported on the union of divisors $\bigcup_{i=1}^l D_i$ (since the results there give a description of the supports of all simple modules in \mathcal{O}_c).

Definition 8.6.1. The cyclotomic Hecke algebra \mathcal{K}_l for $G(1, 1, l)$ is just the commutative algebra $\mathbb{C}[T]$ (where T is the element conventionally denoted by T_0 in the literature) subject to the relation

$$\prod_{i=1}^m (T - q^{a_i}) = 0$$

where $(a_i)_{i=1}^l \in \mathbb{Z}^l$, and $q \in \mathbb{C}^\times$.

To any cyclotomic Hecke algebra one can attach a cyclotomic q -Schur algebra [DJM]. We now recall how it is defined (in our special case). For $i \in \{1, 2, \dots, l\}$ let

$$m_i = \prod_{i < j \leq l} (T - q^{a_j}).$$

and let M^i be the \mathcal{K}_l -ideal $\mathcal{K}_l m_i$. Note that if we let $r_i = \prod_{j=1}^i (T - q^{a_j})$ then $M^i \cong \mathbb{C}[T]/(r_i)$ via the map $f \mapsto f \cdot m_i$. The cyclotomic q -Schur algebra is defined to be

$$\mathcal{S}_l = \text{End}_{\mathcal{K}_l} \left(\bigoplus_{i=1}^l M^i \right) = \bigoplus_{i,j \in [1,l]} \text{Hom}_{\mathcal{K}_l}(M^i, M^j).$$

The following simple lemma yields an explicit description of this algebra, by noting that any \mathcal{K}_l -module can be thought of as a $\mathbb{C}[t]$ -module where t is an indeterminate via the surjection $\mathbb{C}[t] \rightarrow \mathcal{K}_l$ given by $t \mapsto T$.

Lemma 8.6.1. *Let t be an indeterminate, and let g_1 and $g_2 \in \mathbb{C}[t]$. Then we have*

$$\text{Hom}_{\mathbb{C}[t]}(\mathbb{C}[t]/(g_1), \mathbb{C}[t]/(g_2)) \cong \mathbb{C}[t]/(g),$$

where $g = \text{g.c.d.}(g_1, g_2)$.

Proof. If $\phi \in \text{Hom}_{\mathbb{C}[t]}(\mathbb{C}[t]/(g_1), \mathbb{C}[t]/(g_2))$ then $\phi(1 + (g_1)) = u \in \mathbb{C}[t]/(g_2)$, where $g_1 \cdot u = 0 \in \mathbb{C}[t]/(g_2)$. The submodule of $\mathbb{C}[t]/(g_2)$ consisting of such elements is isomorphic as a $\mathbb{C}[t]$ -module to $\mathbb{C}[t]/(\text{g.c.d.}(g_1, g_2))$, and the map $\phi \mapsto u$ gives the required isomorphism. \square

Let $R_i = \text{End}_{\mathcal{K}_l}(M^i) \cong \mathbb{C}[t]/(r_i)$ (by, for example, the previous Lemma). If we write $\mathcal{S}_l^{ij} = \text{Hom}_{\mathcal{K}_l}(M^j, M^i)$ then the Lemma shows that for $i \leq j$ we may view $\text{Hom}_{\mathcal{K}_l}(M^j, M^i)$ as a free $\mathbb{C}[t]/(r_i)$ -module generated by the quotient map $\pi_{ij}: \mathbb{C}[t]/(r_j) \rightarrow \mathbb{C}[t]/(r_i)$ whereas if $i \geq j$ we may view \mathcal{S}_l^{ji} as a free $\mathbb{C}[t]/(r_j)$ -module with generator $m_{ji} = \prod_{i < k \leq j} (t - q^{a_k}) \in \mathbb{C}[t]/(r_i)$, corresponding to the inclusion $\iota_{ij}: \mathcal{K}_l m_j \hookrightarrow \mathcal{K}_l m_i$.

We will thus write an element of S_l as $(a_{ij})_{i,j \in [1,l]}$ where $a_{ij} \in \mathcal{S}_l^{ij}$ is as an element of a free $R_{\min(i,j)}$ -module with generator π_{ji} or ι_{ji} according to whether $i \leq j$ or $i \geq j$ (where we have $\pi_{i,i} = \iota_{i,i} = \text{id}_{M^i}$). If the q^{a_i} s are all distinct, it is straightforward to check that this algebra splits into a direct sum of l matrix algebras of dimensions $1, 2, \dots, l$.

Definition 8.6.2. The algebra S_l has a natural family of modules known as *standard* or *Weyl* modules. For each i, j the algebra R_i has an ideal I_j^i generated by the element

$$s_j^i = \prod_{1 \leq k \leq i; k \neq j} (T - q^{a_k}).$$

If $j \leq i$ this is a one-dimensional \mathbb{C} -vector space, isomorphic to $\mathbb{C}[T]/(T - q^{a_j})$ as an R_i module, otherwise it is zero. For each j , $(1 \leq j \leq l)$ let $W_j = \bigoplus_{k=1}^l I_j^k = \bigoplus_{k=j}^l I_j^k$. We may give W_j the structure of a S_l -module as follows: if $a = (a_{ij})_{1 \leq i, j \leq l} \in S_l$ and $n = (n_k)_{j \leq k \leq l}$ where $a_{ij} \in \mathcal{S}_l^{ij}$, and $n_k \in I_j^k$ then

$$a \cdot n = \left(\sum_r a_{kr} n_r \right)_{j \leq k \leq l},$$

where $a_{kr} n_r$ denotes the image of n_r under the chain of morphisms:

$$R_r \xrightarrow{\cong} M^r \xrightarrow{a_{kr}} M^k \xrightarrow{\cong} R_k.$$

It is easy to see that under this composition, I_j^r maps to I_j^k if $j \leq k$ so that the formula does indeed give an S_l -module structure. Note that in the situation where the q^{a_k} are all distinct, the standard modules are clearly precisely the simple modules.

Example 8.6.3. If $n = 2$ then the algebra S_l can be viewed as the set of matrices of the form

$$\begin{pmatrix} a_1 & a_2 \pi \\ a_3 \varepsilon & b \end{pmatrix}$$

where we write π for π_{21} and $\varepsilon = m_{21}$, and the a_i are in $R_1 = \mathbb{C}$, and $b \in R_2$.

If $q^{a_1} = q^{a_2}$ then setting $\varepsilon = T - q^{a_1}$ we have $R_2 = \mathbb{C}[\varepsilon]$ where $\varepsilon^2 = 0$. Using the basis $\{1, \varepsilon\}$ for R_2 the multiplication in S_2 then becomes:

$$\begin{pmatrix} a_1 & a_2 \pi \\ a_3 \varepsilon & b_1 + b_2 \varepsilon \end{pmatrix} \begin{pmatrix} a'_1 & a'_2 \pi \\ a'_3 \varepsilon & b'_1 + b'_2 \varepsilon \end{pmatrix} = \begin{pmatrix} a_1 a'_1 & (a_1 a'_2 + a_2 b'_1) \pi \\ (a_3 a'_1 + b_1 a'_3) \varepsilon & b_1 b'_1 + \varepsilon (a_3 a'_2 b_1 b'_2 + b'_2 b_1) \end{pmatrix}.$$

It is then easy to see that the subspace of elements of the form $\begin{pmatrix} 0 & 0 \\ a_3 \varepsilon & b_2 \varepsilon \end{pmatrix}$ forms a two-sided ideal, with the quotient algebra isomorphic to the algebra of 2-by-2 upper-triangular matrices.

Let $\rho_i \in S_l$ denote the projection $\rho_i: \bigoplus_{j=1}^l M^j \rightarrow M^i$. We now wish to find a presentation for S_l .

Definition 8.6.4. Let $(a_i)_{i=1}^l \in \mathbb{C}^l$ and let S'_l be the algebra given by generators $T, \rho_i, \pi_{i,i+1}, \iota_{i+1,i}$ subject to relations:

- (1) $\rho_i \rho_j = \delta_{ij} \rho_i, T \rho_i = \rho_i T, \quad (1 \leq i, j \leq l);$
- (2) $\sum_{i=1}^l \rho_i = 1;$
- (3) $\rho_i \pi_{i,i+1} = \pi_{i,i+1} \circ \rho_{i+1} = \pi_{i,i+1}, \quad (1 \leq i \leq l-1);$
- (4) $\rho_{i+1} \iota_{i+1,i} = \iota_{i+1,i} \circ \rho_i = \iota_{i+1,i}, \quad (1 \leq i \leq l-1);$

- (5) $\iota_{i+1,i} \circ \pi_{i,i+1} = (T - q^{a_{i+1}})\rho_{i+1}, \quad (1 \leq i \leq l-1);$
- (6) $\pi_{i,i+1} \circ \iota_{i+1,i} = (T - q^{a_{i+1}})\rho_i, \quad (1 \leq i \leq l-1);$
- (7) $T\rho_1 = q^{a_1}\rho_1.$

It is easy to see that the generator T of S'_l is central, and in fact redundant. (We include it for convenience which will become evident below.)

The next elementary lemma shows moreover that the spectrum of each $T\rho_i$ is very constrained.

Lemma 8.6.2. *Let \mathcal{A} denote an associative $\mathbb{C}(q)$ -algebra with elements A and B , and suppose we set $T_i = AB + q^a$ and $T_{i+1} = BA + q^a$ for some $a \in \mathbb{C}$. If T_i satisfies the equation $P(T_i) = \prod_{k=0}^{i-1} (T_i - q^{a_k}) = 0$, then T_{i+1} satisfies the equation*

$$\prod_{k=0}^i (T_{i+1} - q^{a_k}) = 0.$$

where we set $a_i = a$.

Proof. Clearly we have $Q(AB) = 0$, where $Q(t + q^a) = P(t)$. Now write $Q(t) = \sum_{k=0}^{i-1} b_k t^k$. Then we have

$$\begin{aligned} 0 &= B.Q(AB).A = \sum_{k=0}^{i-1} b_k B.(AB)^k.A, \\ &= \sum_{k=0}^{i-1} b_k (BA)^{k+1} = (BA)Q(BA) \\ &= (T_{i+1} - q^a) \prod_{k=0}^{i-1} (T_{i+1} - q^{a_k}), \end{aligned}$$

since $Q(BA) = P(T_{i+1})$, and thus T_{i+1} satisfies the required equation. \square

Corollary 8.6.1. *The algebra S'_l is isomorphic to the algebra S_l .*

Proof. It is clear that there is a map $p : S'_l \rightarrow S_l$, sending $\pi_{i,i+1}$ and $\iota_{i+1,i}$ to their corresponding elements in S_l and ρ_i to id_{M^i} . Since S_l is generated by the $\pi_{i,i+1}$, $\iota_{i+1,i}$ and id_{M^i} s it is clear that p is surjective. For $i < j$ let $\pi_{ij} = \pi_{i,i+1} \circ \dots \circ \pi_{j-1,j} \in S'_l$, and similarly if $i > j$ let $\iota_{ij} = \iota_{i,i-1} \circ \dots \circ \iota_{j+1,j}$. Then using the defining relations, it is easy to see that as a $\mathbb{C}[T]$ -module S'_l is spanned by the elements $\{\rho_i, \pi_{jk}, \iota_{kj} : 1 \leq i \leq l, 1 \leq j < k \leq l\}$. Applying Lemma 8.6.2 with $T_i = \rho_i T$ and using induction it is easy to see that $\mathbb{C}[T]$ acts on $\mathbb{C}[T]\iota_{kj}$ and $\mathbb{C}[T]\pi_{jk}$ via the algebra $\mathbb{C}[t]/(r_j)$, and on $\mathbb{C}[T]\rho_i$ via the algebra $\mathbb{C}[t]/(r_i)$. Comparing with matrix-like the description of S_l we obtained above it is clear that $\dim_{\mathbb{C}}(S'_l) \leq \dim_{\mathbb{C}}(S_l)$, so that p must be an isomorphism. \square

Remark 8.6.1. It follows that a vector space V can be equipped with an S_l -module structure by giving a grading $V = \bigoplus_{i=1}^l V_i$ together with maps $\alpha_i : V_i \rightarrow V_{i+1}$ and $\beta_i : V_{i+1} \rightarrow V_i$, where we have

$$(8.1) \quad \alpha_{i-1} \circ \beta_{i-1} - \beta_i \circ \alpha_i = (q^{a_{i+1}} - q^{a_i})\text{id}_{V_i}, \quad 1 \leq i \leq l-1,$$

(where we set $\alpha_0 = \beta_0 = 0$).

8.7. We now show that our microlocal approach for the representations of category \mathcal{O}_κ of the rational Cherednik algebra of type μ_l gives a functor to the category of finite dimensional representations of S_l . In fact, using the work of [BK] and [Ku] it follows this functor is an equivalence (provided κ lies outside certain explicit hyperplanes). We use the nearby and vanishing cycle construction for \mathcal{D} -modules on \mathbb{C} which are \mathcal{O} -coherent along \mathbb{C}^\times as described in §7.1. Topologically, this can be viewed as follows: for a \mathcal{D} -module \mathcal{M} , the vector space $\Psi(\mathcal{M})$ corresponds to the stalk of the solution sheaf at a point in the punctured disk, (equipped with the natural monodromy automorphism) and the vector space $\Phi(\mathcal{M})$ gives a “Morse group” of the solution sheaf of \mathcal{M} at a generic covector in the cotangent space $T_0^*\mathbb{C}$, which comes equipped with a microlocal monodromy operator.

Let $\tilde{\mathcal{A}}_{c,i}$ denote the restriction of $\tilde{\mathcal{A}}_c$ to $X_i \cong T^*\mathbb{C}$. It is shown in [Ku, §3] that $\tilde{\mathcal{A}}_{c,i}$ is isomorphic to the standard \mathcal{W} -algebra on $T^*\mathbb{C}$ via the map defined by $x \mapsto f_i$ and $\xi \mapsto g_i$ where we set $f_i = (x_{\eta_1} \dots x_{\eta_i}) \circ (y_{\eta_{i+1}} \dots y_{\eta_l})^{-1}$ and $g_i = (y_{\eta_i} \dots y_{\eta_l}) \circ (x_{\eta_1} \dots x_{\eta_{i-1}})^{-1}$. Note that $f_i \circ g_i = x_{\eta_i} \circ y_{\eta_i}$ and $g_i \circ f_i = x_{\eta_i} \circ y_{\eta_i} + \hbar$. Via this isomorphism, the F -action on $\tilde{\mathcal{A}}_{c,i}|_{X_i}$ corresponds to the F -action on the standard $\mathcal{W}_{T^*\mathbb{C}}$ -algebra given by $x \mapsto t^{2i-l}x$ and $\xi \mapsto t^{l-2i+2}\xi$, so that the F -invariant sections are then:

$$\text{End}_{\text{Mod}_F(\mathcal{W}[\hbar^{1/2}])}(\mathcal{W}[\hbar^{1/2}])^{\text{opp}} = \mathbb{C}[\hbar^{l/2-i}x, \hbar^{i-l/2}\xi],$$

which is isomorphic to $\mathcal{D}(\mathbb{C})$, and moreover the category of modules $\text{Mod}_F(\mathcal{W}[\hbar^{1/2}])$ is equivalent to $\text{Mod}_{\text{coh}}(\mathcal{D}(\mathbb{C}))$ and hence $\text{Mod}_{\text{coh}}(\mathcal{D}_{\mathbb{C}})$ (see the second example of [KR, 2.3.3] for more details, where in our case, $m = 2$). Note that the element $x\partial$ in $\mathcal{D}(\mathbb{C})$ corresponds to $\hbar^{-1}f_i g_i$.

Definition 8.7.1. For $i \in \{1, 2, \dots, l\}$ we define functors KZ_i from \mathcal{O}_c to $\mathbb{C}[t]_{(t)}$ -mod as follows. Let \mathcal{M}_i denote the restriction of \mathcal{M} to X_i , where we may view it (by the above discussion) as a module for $\mathcal{W}_{T^*\mathbb{C}}$, with the appropriate F -action, and hence as a $\mathcal{D}_{\mathbb{C}}$ -module. As such, it corresponds to a holonomic module whose support lies in $\{(x, \xi) : x\xi = 0\}$, where $\{x = 0\}$ corresponds to D_{i-1} and $\{\xi = 0\}$ corresponds to D_i . Therefore it yields a local system on $D_i \setminus \{p_i\} \cong \mathbb{C}^\times$, which is the same data as a vector space equipped with an automorphism, which we may view as a $\mathbb{C}[t]_{(t)}$ -module. We set $KZ_i(\mathcal{M})$ to be this $\mathbb{C}[t]_{(t)}$ -module, $(\Phi(\mathcal{M}_i), T)$.

We now show how the functors KZ_i yield a representation of S_l , where S_l is the algebra with parameters $(q^{a_i})_{i=1}^l$ where $a_i = \sum_{j=1}^{i-1} \bar{c}_j$ (so that $a_0 = 0$) and for $a \in \mathbb{C}$ we write q^a for $\exp(2i\pi a)$. Indeed given \mathcal{M} an object in \mathcal{O}_c , define

$$\mu KZ(\mathcal{M}) = \bigoplus_{i=1}^l KZ_i(\mathcal{M}).$$

Thus $\mu KZ(\mathcal{M})$ is a graded vector space.

Proposition 8.7.1. *The $\mathbb{Z}/l\mathbb{Z}$ -graded vector space $\mu KZ(\mathcal{M})$ has a natural S_l -module structure, so that μKZ becomes a functor from \mathcal{O}_c to S_l -mod.*

Proof. To equip $\mu KZ(\mathcal{M})$ with the structure of a S_l -module, by Remark 8.6.1 we need only define appropriate maps

$$\alpha_i: KZ_i(\mathcal{M}) \rightarrow KZ_{i+1}(\mathcal{M}); \quad \beta_i: KZ_{i+1}(\mathcal{M}) \rightarrow KZ_i(\mathcal{M}).$$

For these we use (slightly rescaled versions of) the natural transformations v and c between the nearby and vanishing cycle functors. Let $\text{var}_i: \Psi(\mathcal{M}_i) \rightarrow \Phi(\mathcal{M}_i)$ and

$can_i: \Phi(\mathcal{M}_i) \rightarrow \Psi(\mathcal{M}_i)$ denote these morphisms, where as above $\mathcal{M}_i = \mathcal{M}|_{X_i}$, and note that $KZ_i(\mathcal{M}) = \Phi(\mathcal{M}_i) = \Psi(\mathcal{M}_{i+1})$.

Set $\alpha_i = q^{a_i} var_{i+1}$ and $\beta_i = can_{i+1}$. We need only verify that Equation 8.1 holds. For this, note that if T_i denotes the monodromy automorphism on $KZ_i(\mathcal{M})$, then $T_i = \exp(-2i\pi\hbar^{-1}f_i g_i) = var_i can_i + 1$. Since on $X_i \cap X_{i+1}$ we have $f_i g_i = x_{\eta_i} \circ y_{\eta_i} = x_{\eta_{i+1}} \circ y_{\eta_{i+1}} + \hbar \bar{c}_i$, it follows that $T_i = q^{\bar{c}_i} T_{i+1}$ on $KZ_i(\mathcal{M}) = \Phi(\mathcal{M}_i) = \Psi(\mathcal{M}_{i+1})$. Then we have

$$\begin{aligned} \alpha_{i-1} \circ \beta_{i-1} - \beta_i \circ \alpha_i &= q^{a_i}(T_i - 1) - q^{a_{i+1}}(T_{i+1} - 1) \\ &= q^{a_i}(q^{\bar{c}_i} T_{i+1} - 1) - q^{a_{i+1}}(T_{i+1} - 1) \\ &= q^{a_{i+1}} - q^{a_i}, \end{aligned}$$

as required. Finally, note that on X_1 the module \mathcal{M}_1 is supported entirely on D_1 , so that $\Psi(\mathcal{M}_1) = 0$, and hence the above calculation also shows that $\beta_1 \circ \alpha_1 = q^{a_2} - q^{a_1}$ and we are done. \square

Corollary 8.7.1. *Let*

$$s_i(t) = \prod_{j=1}^i (t - q^{\sum_{k=j}^{i-1} \bar{c}_k}).$$

The functor KZ_i from \mathcal{O}_c to $\mathbb{C}[t]_{(t)}$ factors through $\mathbb{C}[t]/(s_i)$.

Proof. If T is the central element of S_l , then the action of the monodromy operator on $KZ_i(\mathcal{M})$ (for \mathcal{M} an object of \mathcal{O}_c) is given by $q^{\sum_{k=i}^l \bar{c}_k} \rho_i T$. The claim then follows from fact our description of the structure of S_l . \square

Remark 8.7.1. Note that in particular the functor KZ_l therefore yields a representation of the corresponding cyclotomic Hecke algebra. In fact it is just the original KZ -functor for the rational Cherednik algebra of type μ_l . This can be seen using the correspondence between the κ_i and $c_i s$, and an argument analogous to Proposition 5.5.1. Using [Ku] (where an explicit construction of the sheaves corresponding to standard modules is given), the reader can check that μKZ sends the standard modules in \mathcal{O}_c to the standard modules for S_l , and similar arguments allow one to show that μKZ is in fact an equivalence.

9. APPENDIX

9.1. Twisted \mathcal{D} -modules. In this paper we work with modules over rings of twisted differential operators on $\mathfrak{gl}_n \times \mathbb{P}^{n-1}$, so we recall briefly the construction of the twisted rings we use. For a detailed discussion of these issues see [K7], whose presentation we largely follow¹⁰ (for an alternative account in the algebraic context see [BB]). If X is a topological space a *twisting datum* τ is a triple $(\pi: X_0 \rightarrow X, L, m)$ where $\pi: X_0 \rightarrow X$ is a continuous map admitting a section locally on X , L is an invertible sheaf of vector spaces on $X_0 \times_X X_0$ and m is an isomorphism:

$$m: p_{12}^{-1} L \otimes p_{23}^{-1} L \rightarrow p_{13}^{-1} L,$$

on $X_2 = X_0 \times_X X_0 \times_X X_0$. Moreover, the isomorphism m is required to satisfy an appropriate “associativity” condition on the quadruple product of X_4 of X_0 over

¹⁰As noted in [K7], the presentation there uses the notion of a twisting datum (recalled above) which is slightly *ad hoc* from a systematic point of view, however it is well-suited to the families of twistings which we need.

X . Given a twisting datum, a twisted sheaf on X is a pair (F, β) consisting of a sheaf F on X_0 and an isomorphism $\beta: L \otimes p_2^{-1}F \rightarrow p_1^{-1}F$, where p_1, p_2 are the natural projections from $X_1 = X_0 \times_X X_0$ to X_0 , along with the requirement that β satisfies an appropriate cocycle condition.

If H is a complex affine algebraic group and $\pi: X_0 \rightarrow X$ a principal H -bundle (where we assume that H acts on X_0 on the left) then one can naturally attach to X_0 a family of twisting data on X . To describe this we first need the notion of a character local system. Let $\mu: H \times H \rightarrow H$ be the multiplication map. A character local system is an invertible \mathbb{C}_H -sheaf L equipped with an isomorphism $m: q_1^{-1}L \otimes q_2^{-1}L \rightarrow \mu^{-1}L$ which satisfies the associative law. Let $\mathfrak{h} = \text{Lie}(H)$ denotes the Lie algebra of H and take an H -invariant element λ of $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$, so that $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$. Let L_λ be the sheaf of (analytic) functions on H which satisfy $R_H(X)(f) = \lambda(X).f$, where R_H denotes map from \mathfrak{h} to vector fields on H given by the right action of H on itself. The multiplication map $H \times H \rightarrow H$ then induces the structure of a character local system on L_λ .

Now suppose that in addition we have an H -space Y , and let $a: H \times Y \rightarrow Y$ be the action map, and $p_1: H \times Y \rightarrow Y, p_2: H \times Y \rightarrow H$ the obvious projections. An (H, λ) -equivariant sheaf on Y , or λ -twisted equivariant sheaf, is a pair (F, β) consisting of a sheaf F on Y together with an isomorphism $\beta: p_1^{-1}L_\lambda \otimes p_2^{-1}F \rightarrow a^{-1}F$, once again satisfying an appropriate associative condition. In the case when $\lambda = 0$ we say that F is an equivariant sheaf. The notion of a twisted equivariant sheaf is closely related to that of a twisted sheaf, as the following construction shows.

Let $\pi: X_0 \rightarrow X$ be a principal H -bundle, and let λ be as above. We may identify $H \times X_0$ with $X_0 \times_X X_0$ by the map $(h, x') \mapsto (hx', x')$ and via this isomorphism, we find that $p_1^{-1}L_\lambda$ yields a twisting datum τ_λ on X . In particular, the categories of τ_λ -twisted sheaves on X is equivalent to the category of (H, λ) -equivariant sheaves on X_0 , and thus the principal bundle X_0 yields the family $\{\tau_\lambda : \lambda \in (\mathfrak{h}^*)^H\}$ of twisting data.

Thus far our discussion has been purely topological, but the same formalism may be used with \mathcal{D} -modules as is explained in [K7, §7.11]. One replaces L_λ with the sheaf $\mathcal{L}_\lambda = \mathcal{D}_H.u_\lambda$ where u_λ has defining relations $R_H(A)u_\lambda = \lambda(A)u_\lambda$ for $A \in \mathfrak{h}$ (and pullbacks etc. of sheaves of vector spaces with their appropriate \mathcal{D} -module analogues). Note that $L_\lambda \cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}_\lambda, \mathcal{O}_H^{\text{an}})$, where $\mathcal{O}_H^{\text{an}}$ is the sheaf of analytic rather than algebraic functions on H , and in fact the Riemann-Hilbert correspondence extends to have a twisted analogue. The ring of twisted differential operators $\mathcal{D}_{X, \tau_\lambda}$ (or more simply $\mathcal{D}_{X, \lambda}$) on X is then given as follows: if $\mathcal{N}_\lambda = \mathcal{D}_{X_0}v_\lambda$ is the \mathcal{D}_{X_0} -module with defining relations $L_{X_0}(A) = -\lambda(A).v_\lambda$, then

$$\mathcal{D}_{X, \lambda} = \{f \in \pi_*(\mathcal{E}nd_{\mathcal{D}_{X_0}}(\mathcal{N}_\lambda)) : f \text{ is } H\text{-equivariant}\}^{\text{op}}$$

where $(\cdot)^{\text{op}}$ denotes the opposite ring. By [K7, Lemma 7.12.1], the category of modules for the τ_λ -twisted ring of differential operators $\mathcal{D}_{X, \lambda}$ is then equivalent to the category of (H, λ) -equivariant \mathcal{D}_{X_0} -modules on X_0 . Note that in the terminology of [K7], (H, λ) -equivariant \mathcal{D} -modules form an abelian subcategory of the category of H -quasi-equivariant \mathcal{D} -modules.

Finally, we need to consider equivariant twisted \mathcal{D} -modules. Given an affine algebraic group G and a G -space X , one can naturally define the notion of a G -equivariant twisting datum on X , and hence the notion of G -equivariant twisted

sheaves on X . In particular, in the case where the twisting data arises from a principal H -bundle X_0 , if the group G acts on X_0 and X in such a way that the map π is G -equivariant, and the actions of H and G commute on X_0 , then we may define G -equivariant twisted $\mathcal{D}_{X,\lambda}$ -modules, and the equivalence between twisted modules on X and (H, λ) -equivariant modules on X_0 allows us to identify such modules with \mathcal{D}_{X_0} -modules which are G -equivariant and (H, λ) -twisted equivariant.

9.2. Index Theorems and Characteristic Cycles. We have computed the characteristic cycle of standard modules using the local Euler obstruction and the known calculation of the stalk cohomologies. In this appendix we briefly recall the index theorem of Kashiwara, Dubson, Brylinski [K1], [BDK], and in the topological setting [M] which is the key to this approach. This theorem can be seen as a (local version of a) generalization of the classical Hopf index theorem which calculates the Euler characteristic of a compact manifold X as the self-intersection number of X in its cotangent bundle.

Since we only need a local result, we may suppose that X is a stratified analytic subset of affine space \mathbb{C}^n . Thus $X = \bigsqcup_{S \in \mathcal{S}} S$, where each S is a smooth locally-closed connected subset of \mathbb{C}^n , and the closure of a stratum S is a union of strata. We may also assume that the Whitney (a) and (b) conditions are satisfied (or for that matter the μ -condition introduced by Kashiwara and Schapira [KS, Chapter 8]). For $x \in X$ we will write $B_\varepsilon(x)$ for the set

$$\{y \in X : \|y - x\| < \varepsilon\},$$

where $\|\cdot\|$ is the standard Hermitian norm on \mathbb{C}^n . (Thus our constructions here use the analytic variety attached to the algebraic varieties we considered earlier).

Definition 9.2.1. Let $z \in S$ be a point of the stratum S . By taking a normal slice N to S at z , that is, choosing a complex analytic submanifold N which intersects each stratum transversely and such that $N \cap S = \{z\}$, we may reduce to the case $S = \{z\}$. Let $\phi: N \rightarrow \mathbb{C}$ be a holomorphic function vanishing at z such that $d\phi(z)$ does not lie in the closure of T_T^*X for any stratum $T \neq S$. If $S = T$ we define $c_{S,S} = 1$ for all S .

Endow X with a Hermitian metric (say by taking the restriction of the standard on \mathbb{C}^n) and pick a small disk $B = B_\varepsilon(z)$ about z , and a generic $\eta \in \mathbb{C}^\times$ such that $|\eta| \ll \varepsilon$. The complex link L of the stratum S in T is then defined to be the set

$$L = B \cap T \cap N \cap \phi^{-1}(\eta).$$

Stratified Morse theory shows that the homeomorphism type of L is independent of the choices made (in fact it is independent of the metric $\|\cdot\|$ also, so does not depend on the choice of local embedding we make). We define the local Euler Obstruction $c_{S,T}$ to be the Euler characteristic of the complex link, that is

$$c_{S,T} = \chi_c(L).$$

As the notation suggests, this number is independent of the choice of z in the stratum S (here we use the assumption that our strata are connected).

The index theorem shows that the characteristic cycle of a holonomic \mathcal{D} -module determines its local Euler characteristics. Indeed suppose that M is a \mathcal{D} -module whose characteristic cycle lies in $\bigsqcup_{S \in \mathcal{S}} T_S^*M$ (so in particular M is holonomic).

Then since we assume that our stratification satisfies the Whitney conditions, $\mathrm{DR}(M)$ is a constructible sheaf which is locally constant on the strata S , and we may set

$$\chi_S(M) = \sum_i (-1)^i \dim(\mathcal{H}^i(\mathrm{DR}(M))|_S).$$

We also have $CC(M) = \sum_{S \in \mathcal{S}} m_S(M) [T_S^* X]$.

Theorem 9.2.1. *Let M be a holonomic \mathcal{D} -module as above. Then we have*

$$m_S(M) = \sum_{T \subset \bar{S}} c_{S,T} \chi_T(M).$$

Remark 9.2.1. Since $c_{S,S}$ is defined to be 1, if we pick any total order refining the partial order on strata give by the closure relation we see that the matrix $(c_{S,T})$ is unitriangular, and so the sets of numbers $\{\chi_S(M)\}_{S \in \mathcal{S}}$ and $\{m_S(M)\}_{S \in \mathcal{S}}$ determine each other. Thus one can invert the above theorem to give a formula for the multiplicities of the characteristic cycle in terms of the local Euler characteristics. This is the form of the theorem stated in [K1]. In fact the map from holonomic \mathcal{D} -modules to constructible functions given by the χ_S and the characteristic cycle map from holonomic \mathcal{D} -modules to Lagrangian cycles induce isomorphisms from the Grothendieck group to the group of constructible functions and Lagrangian cycles respectively. The index theorem is then an explicit geometric description of the induced isomorphism between the two abelian groups. Note that the difference in signs between our statement of the index theorem and that in [EM] arises as we work with \mathcal{D} -modules rather than constructible sheaves.

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